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A small dual-automorphic lattice with no involutory dual automorphism

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Abstract

An exercise in Garrett Birkhoff's renowned book on lattice theory asks for a lattice with 18 elements and of length 5 that has a dual automorphism, but no involutory dual automorphism. This note constructs a smaller lattice, 15 elements and length 4, with the same property.

Garrett Birkhoff's renowned text *Lattice Theory*, revised edition, has the following exercise on page 22:

> Ex. 6*. Find a lattice of length 5 and 18 elements which has a dual automorphism, but no involutory dual automorphism.

A dual automorphism of a lattice $(L, \leq)$ is a permutation f of L such that

$$\langle \forall x, y \mid x \in L \wedge y \in L \triangleright \quad f.x \leq f.y \quad \equiv \quad y \leq x \rangle \qquad .$$

For example, one dual automorphism of the lattice depicted in Figure 0(a) is the function f defined by

$$f.a = e \quad f.b = c \quad f.c = b \quad f.d = d \quad f.e = a \qquad . \tag{0}$$

A dual automorphism is *involutory* if its square is the identity, that is, if it is its own inverse. For example, the dual automorphism (0) is involutory. Birkhoff's exercise asks us to construct a small lattice with a dual automorphism, but no involutory dual automorphism. Call such a lattice a DANIDA lattice.

The authors have not been able to find the DANIDA lattice that Birkhoff had in mind, but they have found a smaller one (15 elements and length 4). The rest of this note describes that lattice.

We're looking for a small DANIDA lattice, a finite one. The empty lattice isn't a DANIDA lattice, because its only dual automorphism is involutory, and ditto for the lattice of one element. Hence, the lattice we're looking for has a bottom and a top, call them $\bot$ and $\top$, with $\bot \leq x$ and $x \leq \top$ for every lattice element x.

A dual automorphism of a lattice with a bottom and a top must swap them, and hence cannot have odd order. Order 2 means involutory, and the smallest even number greater than 2 is 4, so let's aim for a dual automorphism of order 4.

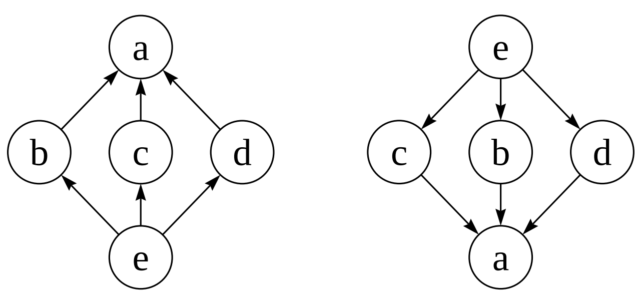

Figure 0: (a) Example lattice, (b) the lattice transformed by the map f in (0)

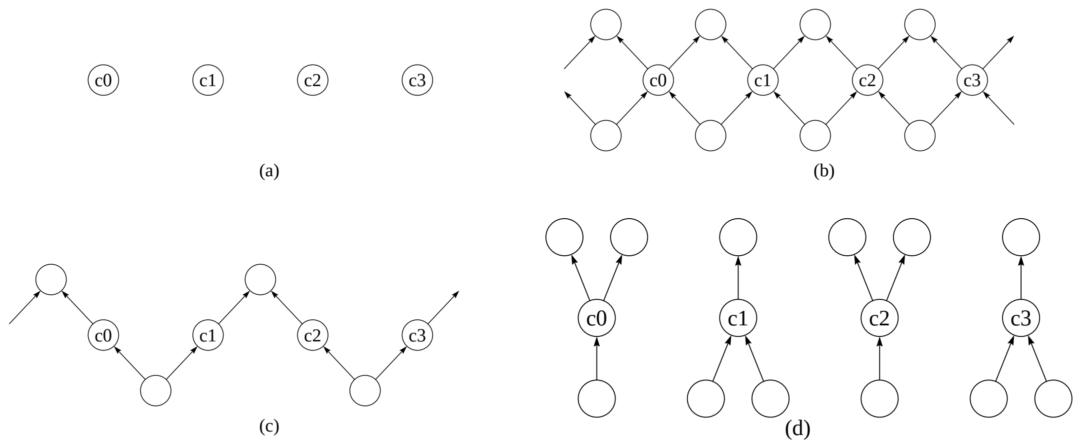

Figure 1: Ideas: (a) four elements at the same height, (b) example cyclic arrangement of the four elements, (c) cyclic arrangement that alternates through down- and up-neighbors, (d) alternating (up, down)-degrees

Our first idea is to add four elements to the lattice in such a way that any dual automorphism must permute the four elements. We do that by making the four elements the only elements in the lattice at their height, each placed equally far from bottom as from top. Figure 1(a) shows four elements, called c_0, c_1, c_2, and c_3, placed at the same height. We must make sure that the permutation of the four elements in any dual automorphism is not involutory. To that end, we can cut down the 24 possible permutations to 8 by arranging the elements cyclicly, for example as shown in Figure 1(b). (The dangling edges at the sides of the figure indicate edges that wrap around.) We can further reduce the number of possible permutations from 8 to 4 by letting the cycle alternate through down-neighbors and up-neighbors, as in Figure 1(c). Finally, we reduce the number of permutations from 4 to 2 by making the up-degree of an element different from its down-degree, an idea shown in isolation in Figure 1(d). We show a combination of these ideas in Figure 2.

Let's check that Figure 2 is a DANIDA lattice. First, note that all four of the c elements are the same distance from bottom as from top. Thus, any dual automorphism permutes the c elements. Similarly, only the e elements have up- and down-degree 1, so any dual automorphism permutes the e elements. Consequently, any dual automorphism also permutes the remaining elements, the d elements.

Because the up- and down-degrees of each c element must be swapped by a dual automorphism, any dual automorphism maps c_i to either c_{i+1} or c_{i-1}, but not c_i or

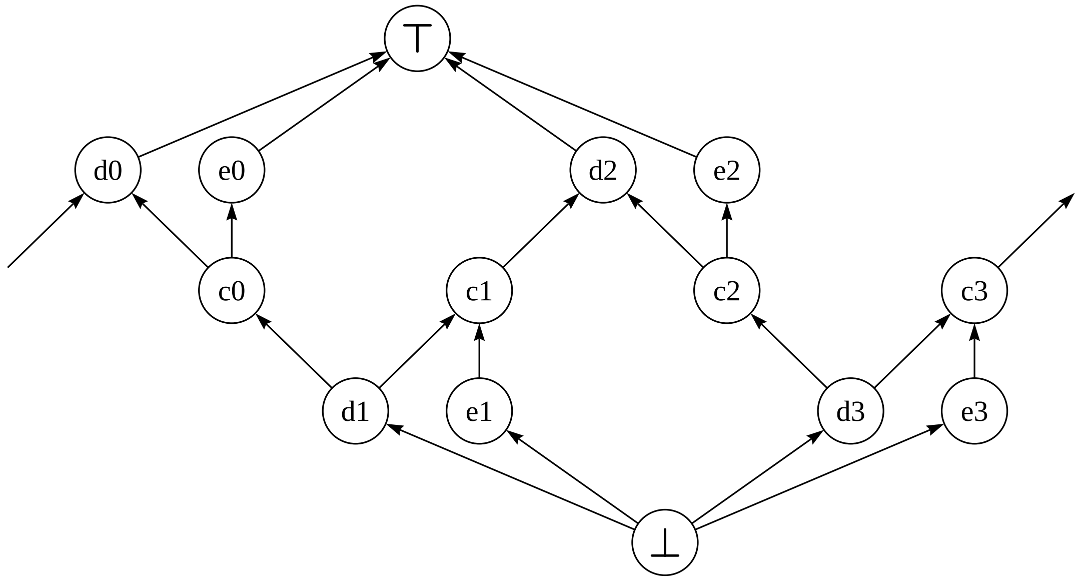

Figure 2: The four elements c_0, c_1, c_2, and c_3, cyclicly arranged and with alternating (up, down)-degrees of (2,1) and (1,2)

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c_{i+2}. (The indices are modulo 4.) Because each e element neighbors a particular c element, any dual automorphism that maps c_i to c_{i+k} also maps e_i to e_{i+k}. Finally, let's look at the d's, using f to denote a dual automorphism that satisfies $f.c_0 = c_1$. Since d_1 is the only down-neighbor of c_0, the element $f.d_1$ must be the only up-neighbor of $f.c_0$, so $f.d_1 = d_2$. Furthermore, since element d_1 has two up-neighbors, c_0 and c_1, element $f.d_1$ (that is, d_2) must have two down-neighbors, $f.c_0$ and $f.c_1$. Since the two down-neighbors of d_2 are c_1 and c_2, and we have already determined that $f.c_0 = c_1$, we conclude that $f.c_1 = c_2$. By symmetry, we are now done: there are precisely two dual automorphisms,

$$f.c_i = c_{i+1} \qquad f.d_i = d_{i+1} \qquad f.e_i = e_{i+1}$$

and

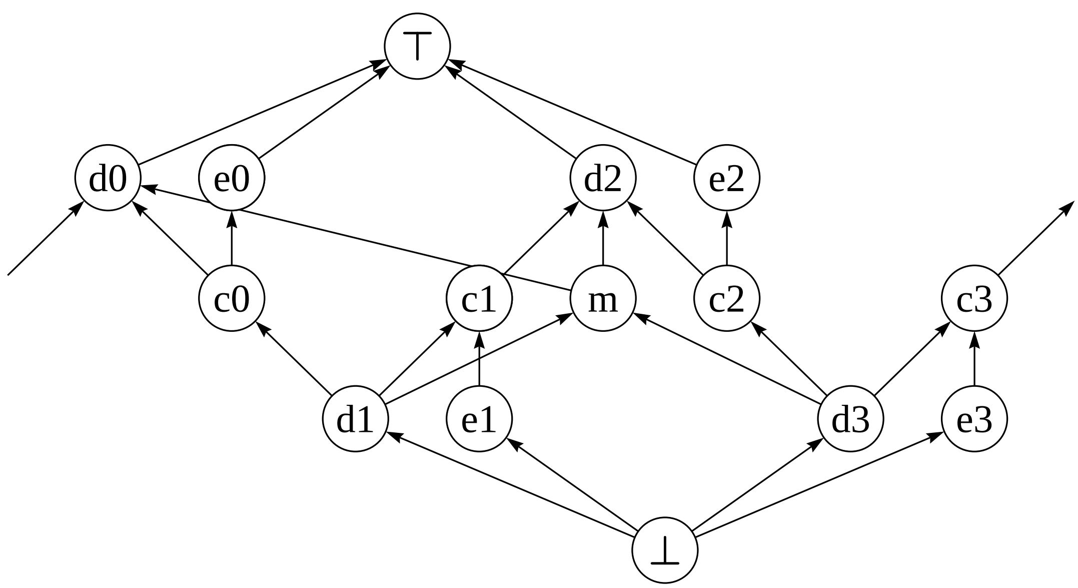
$$g.c_i = c_{i-1} \qquad g.d_i = d_{i-1} \qquad g.e_i = e_{i-1} \qquad ,$$

and neither is involutory.

Actually, we're not quite done. The result is supposed to be a lattice, so every pair of elements x and y must have a join $x \uparrow y$ and a meet $x \downarrow y$. By symmetry, it suffices to check that joins exist. If either of the two elements under consideration is bottom or top, the join exists, for $\bot \uparrow x = x$ and $\top \uparrow x = \top$; so we restrict our attention to non-extreme elements. Consider an element x whose upper bounds form a chain, that is, are totally ordered. Any nonempty set of upper bounds for such an element x has a least element; so the join $x \uparrow y$ exists, for all y. Hence, it suffices to check all possible pairs of the four elements c_0, d_1, c_2, and d_3. Unfortunately, such a check quickly reveals that what we have is not a lattice: the common upper bounds of d_1 and d_3 are $\top$, d_0, and d_2, and there is no minimum among those three elements.

Luckily, we can restore the lattice property without ruining the dual-automorphism properties. We do so by adding an element m, which we place below d_0 and d_2 and above d_1 and d_3. This restores the lattice property, because the joins of m with each of the elements c_0, d_1, c_2, and d_3 exist (they are d_0, m, d_2, and m, respectively). A dual automorphism has no choice but to fix m, that is, map m to itself, because m is the only element whose up- and down-degrees are both 2. Hence, we have not introduced any new dual automorphisms. Furthermore, the dual automorphisms f and g remain dual automorphisms if we extend them to fix m, because $m \leq d_i$ just when $f.d_i = d_{i+1} \leq m = f.m$, and similarly for g.

The result is a DANIDA lattice with 15 elements and of length 4:



The lattice can be nicely rendered in 3 dimensions, as shown in Figure 3. Note that the dual automorphisms are there realized as rotary reflections in the horizontal mid-plane. A 3-D animation of the lattice can be found from `http://www.research.digital.com/SRC/publications/src-tn.html`, report 1997-008. (Thanks, Marc Najork!)

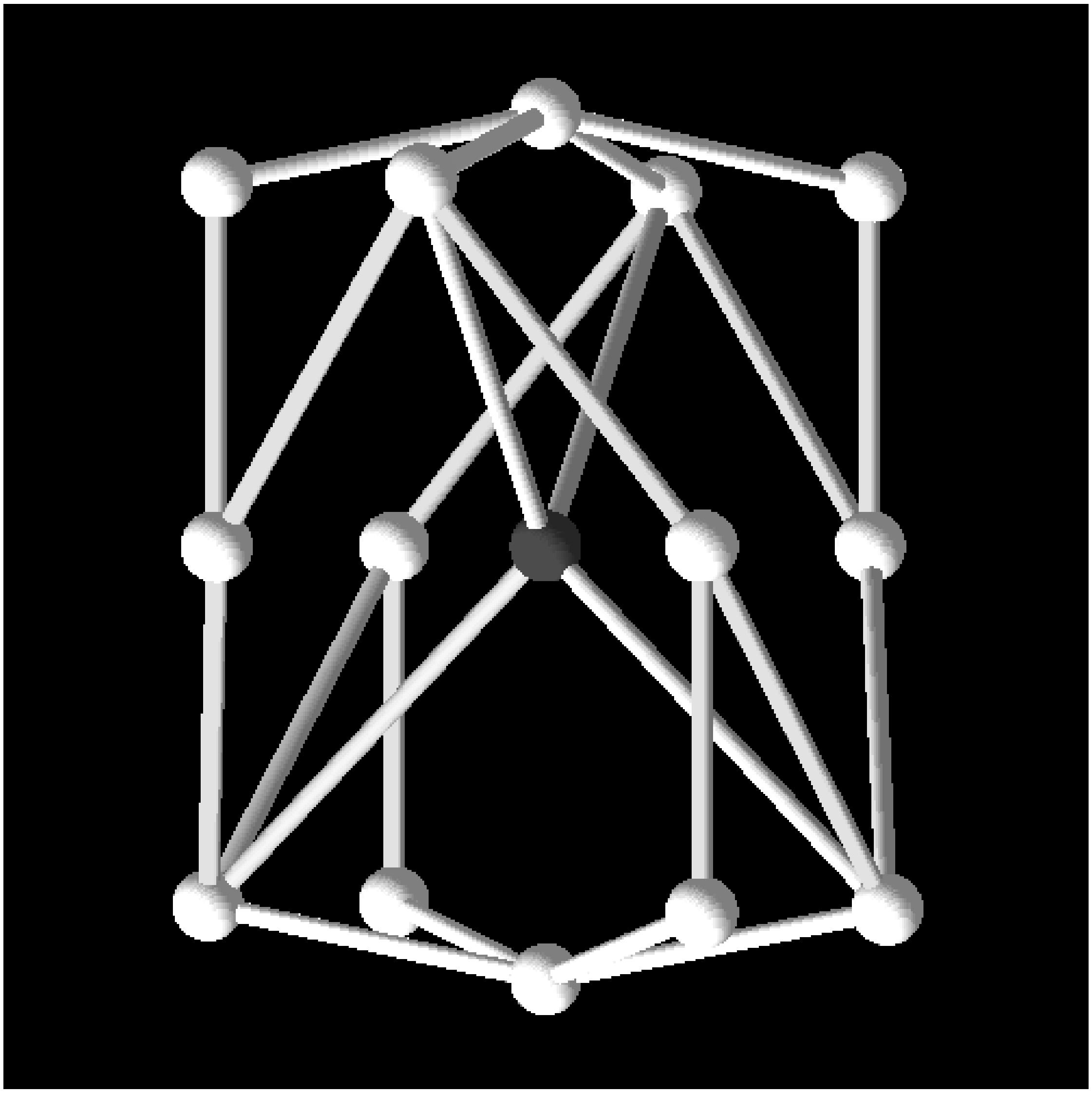


Figure 3: The final lattice, rendered in 3 dimensions