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Primitives for Computational Geometry

by Jorge Stolfi

January 27, 1989

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Robert W. Taylor, Director

Primitives for Computational Geometry

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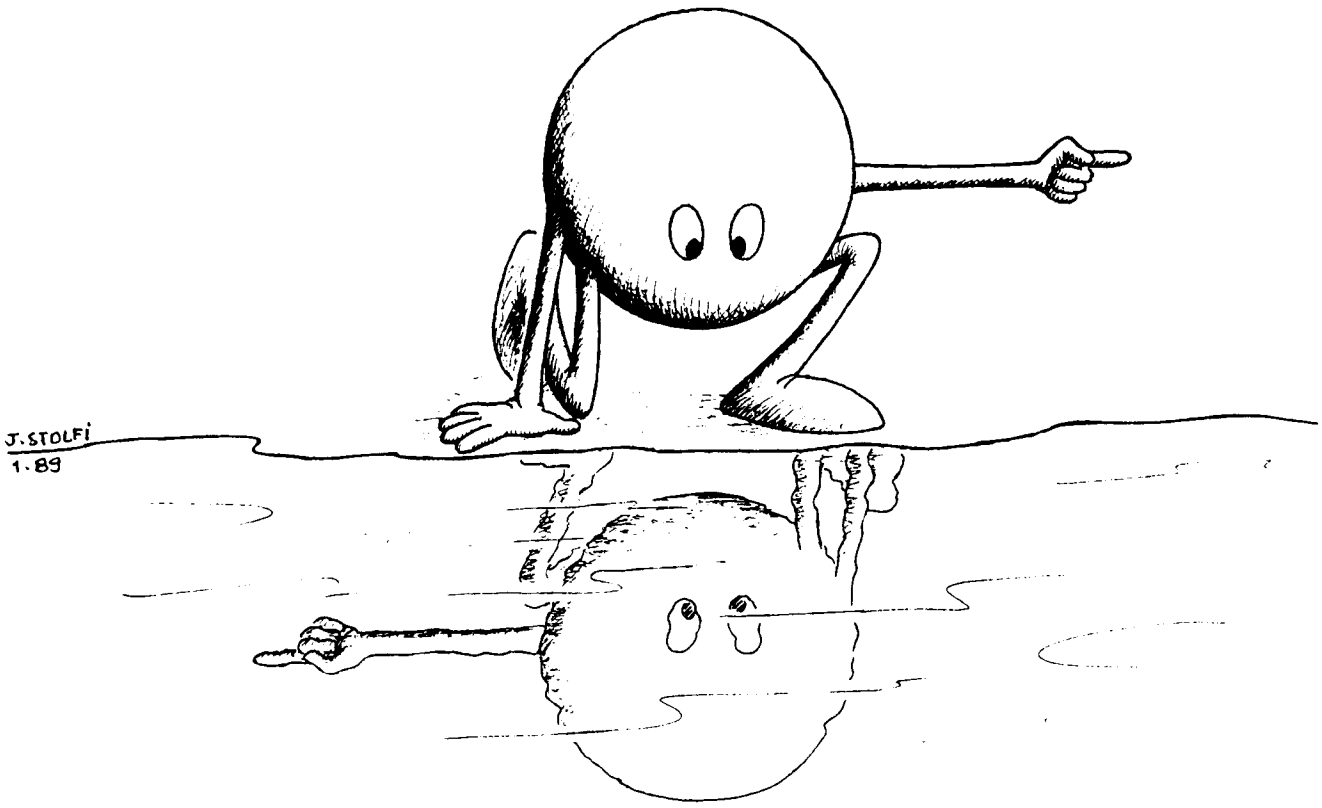

Publication history

This report reproduces a dissertation submitted in May 1988 to the Department of Computer Science of Stanford University in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Author's abstract

Many geometric algorithms become simpler, more general, and more efficient when recast in the language of projective geometry. Some reasons for this are the uniform handling of points at infinity, the attendant reduction in the number of special cases, and the perfect duality between points and hyperplanes that are possible in the projective model. In fact, the homogeneous coordinates so widely used in computer graphics are essentially an analytical model of classical projective geometry.

However, projective space is topologically quite different from Euclidean space. For example, in the projective plane lines have only one side, all triangles have the same handedness, and there are two distinct segments with any given pair of endpoints. These differences are a serious practical problem, since many geometric algorithms depend on orientation, ordering and separation tests that make sense only in the Euclidean model.

This dissertation describes a slightly modified form of projective geometry which is free from this problem. Analytically, the change consists in making the signs of homogeneous coordinates more significant. Geometrically, the change consists in adopting *oriented* lines and planes as the elementary objects of the model, and redefining the basic geometric operation of meet and join so as to produce results with a definite orientation. Topologically, this is equivalent to working with a double covering projective space, which is equivalent to an n-dimensional sphere.

The resulting framework, here called *oriented projective geometry*, combines the elegance of classical projective geometry with the ability to talk about oriented lines and planes, signed angles, line segments, convex figures, and many other concepts that cannot be conveniently defined within that model. The goals of this dissertation are: (1) to develop an intuitive understanding of oriented projective geometry in two and three dimensions; (2) to describe a formal geometric calculus for handling oriented lines, planes, and flat spaces of arbitrary dimension; and (3) to investigate the efficient representation of such objects in computers.

Jorge Stolfi

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Chapter 1
Introduction

Oriented projective geometry is a model for geometric computation that combines the elegance of classical projective geometry with the ability to talk about oriented lines and planes, signed angles, line segments, convex figures, and many other concepts that cannot be defined within the classical version. Classical projective geometry is the implicit framework of many geometric computations, since it underlies the homogeneous coordinate representation widely used in computer graphics. It is argued here that *oriented* projective geometry — and its analytic model, based on *signed* homogeneous coordinates — provide a better foundation for computational geometry than their classical counterparts.

The differences between the classical and oriented versions are largely confined to the mathematical formalism and its interpretation. Computationally, the changes are minimal and do not increase the cost and complexity of geometric algorithms. Geometric algorithms that use homogeneous coordinates can be easily converted to the oriented framework at little cost. The necessary changes are largely a matter of paying a little more attention to the order of operands and to the signs of coordinates.

The aim of this thesis is not so much to advance the remote frontiers of pure geometry or theoretical computer science, but rather to assemble a rich, consistent, and effective set of basic tools for computational geometry that can be used by programmers in their everyday work. Its novelty lies not in the mathematical concepts and algorithms, but in their application to the world of practical computing. Mathematicians will not find in this thesis any deep theorems or revolutionary definitions; computer scientists will search in vain for powerful algorithms or sophisticated data structures. The specialists will notice that oriented projective geometry is essentially equivalent to spherical (or double elliptic) geometry, which to them is an old and well-explored concept.[1] However, programmers will (hopefully) be surprised on learning that the geometry of the sphere, an eminently curved surface with curved lines, is in fact an excellent model for ordinary geometric computations with straight lines on the flat Euclidean plane.

My goal is to make programmers and computational geometers aware of this connection, and give them a set of intellectual tools to help them exploit it, including a firm intuition and a practical notation. With this in mind, I have strived to keep mathematical jargon and formalism to a minimum, and substitute intuition

for formalism whenever possible. At the risk of being tedious, I have also tried to illustrate general definitions and theorems with examples in one, two, and three dimensions.

Here is a brief outline of the thesis. The rest of this chapter contains a brief description of the classical and oriented projective planes, their advantages and disadvantages. Chapters 2 through 7 describe the canonical oriented projective spaces (of arbitrary dimension) in detail, defining their subspaces (points, lines, planes, etc.), orientations, the fundamental geometric operations of join and meet, and the concept of relative orientation or "sideness." Chapter 8 defines projective maps, the transformation of space that preserve incidence and orientation. These are used in chapter 9 to define abstract oriented projective spaces. Chapter 10 introduces the notion of projective duality, which allows us to exchange join with meet. Chapters 11, 12, and 13 discuss concepts related to projective maps, such as projective functions, projective frames and coordinate systems, and the cross-ratio of four points. Chapter 14 shows how oriented projective geometry allows us to define convexity in a way that preserves its most important properties. In chapters 15, 16, and 17 we will see how to emulate the affine, Euclidean, and linear vector spaces within the oriented projective framework. Finally, chapters 18 through 20 discuss the representation and manipulation of lines, planes, etc. in the computer.

1. Classical projective geometry

Before we go on, a short review of classical projective geometry may be in order. The classical projective plane may be defined by means of four mathematical models (*straight*, *spherical*, *analytic*, and *homogeneous*).[7] It is also possible to define the projective plane abstractly, as a set of objects satisfying certain axioms.[3,4] Unfortunately, such axiomatic definitions are hard to to generalize to higher dimensions. Moreover, the axiomatic method is better at formalizing intuitive knowledge than at developing intuition about a new subject. Considering the aims of this thesis, I have opted to base all definitions on concrete models, without trying to abstract from them a coherent set of axioms.

1.1. The straight model

The *straight model* of the projective plane $\mathbf{P}_2$ consists of the real plane $\mathbf{R}^2$, augmented by a *line at infinity* Ω, and by an *infinity point* $d\infty$ for each pair of opposite directions $\{d, -d\}$. The point $d\infty = (-d)\infty$ is by definition on the line Ω

and also on every line that is parallel to the direction d. See figure 1.

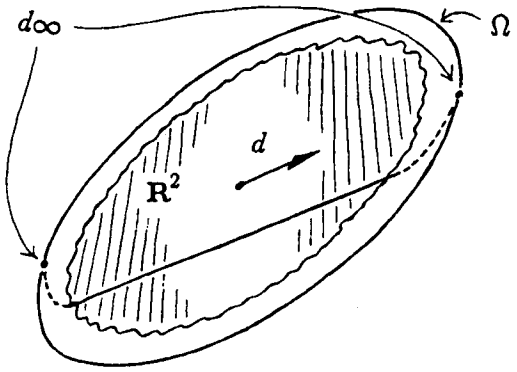

Figure 1. The straight model of the projective plane $\mathbf{P}_2$.

1.2. The spherical model

The *spherical model* of $\mathbf{P}_2$ consists of the surface of a sphere, with diametrally opposite points identified. The lines of $\mathbf{P}_2$ are represented by the great circles of the sphere, again with opposite points identified. See figure 2.

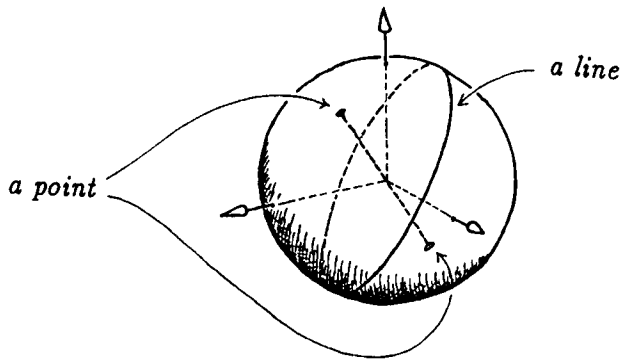

Figure 2. The spherical model.

The spherical model clearly shows that all lines and points are equivalent in their topological and incidence properties. The seemingly special character of Ω and the infinite points in the straight model is a mere artifact of the representation.

1.3. The analytic model

The *analytic model* represents points and lines of $\mathbf{P}_2$ by their *homogeneous coordinates.* A point is by definition a non-zero triplet of real numbers $[w, x, y]$, with scalar multiples identified. By this we mean $[w, x, y]$ and $[\lambda w, \lambda x, \lambda y]$ are the same point, for all $\lambda \neq 0$. A line is also represented by a non-zero real triplet $\langle W, X, Y \rangle$, which by definition is incident to all points $[w, x, y]$ such that $Ww + Xx + Yy = 0$. Note that $\langle W, X, Y \rangle$ and $\langle \lambda X, \lambda Y, \lambda Z \rangle$ are the same line for all $\lambda \neq 0$.

1.4. The homogeneous model

Geometrically, we can identify the point $[w, x, y]$ of $\mathbf{P}_2$ with the line of $\mathbf{R}^3$ passing through the origin and through the point (w, x, y). The line $\langle W, X, Y \rangle$ of $\mathbf{P}_2$ then corresponds to the plane of $\mathbf{R}^3$ passing through the origin and perpendicular to the vector (W, X, Y). This is the *homogeneous model* of $\mathbf{P}_2$. See figure 3.

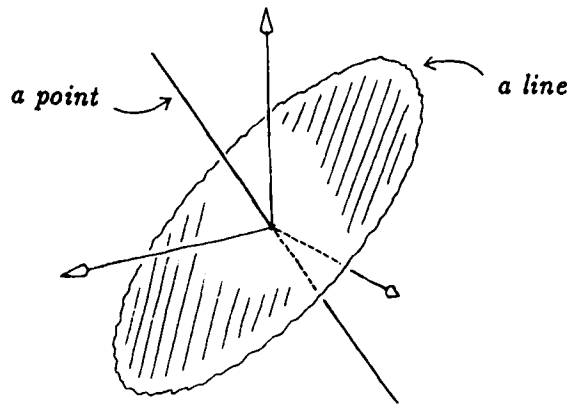

Figure 3. The homogeneous model of $\mathbf{P}_2$.

1.5. Correspondence between the models

The analytic and straight models of $\mathbf{P}_2$ are connected by the familiar homogeneous-to-Cartesian coordinate transformation, whereby the homogeneous triplet $[w, x, y]$ is mapped to the point $(x/w, y/w)$ of the Cartesian plane. We can view this transformation as choosing among all equivalent homogeneous triplets a *weight-normalized* representative $(1, x/w, y/w)$ (the first coordinate w being called the *weight* of the triplet). Homogeneous triplets with $w = 0$ correspond to the infinity points of the straight model. A triplet $[w, x, y]$ corresponds also to the point

$$(w, x, y)/\sqrt{w^2 + x^2 + y^2}$$

of the spherical model.

Geometrically, these mappings corresponds to *central projection* of $\mathbf{R}^3$ onto the unit sphere, or onto the plane π tangent to the sphere at $(1, 0, 0)$. See figure 4. This projection takes a pair of diametrally opposite points p, p' of the sphere to the point q where the line pp' meets the tangent plane π. The great circle of the sphere that is parallel to the plane π is by definition projected onto the line at infinity Ω of the straight model. Observe how this correspondence preserves points, lines, and their incidence relationships.

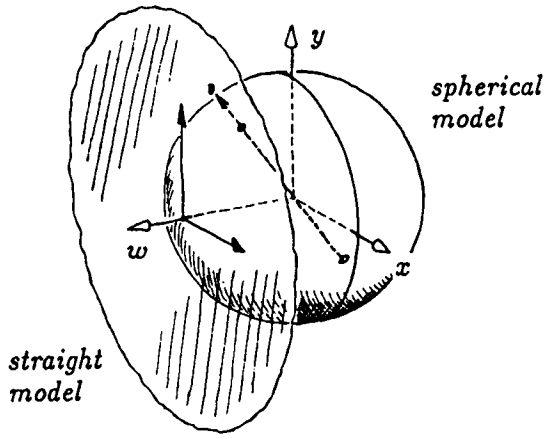

Figure 4. Central projection between the models of $\mathbf{P}_2$.

2. Advantages of projective geometry

2.1. Simpler formulas

Projective geometry and homogeneous coordinates have many well-known advantages over their Cartesian counterparts. For one thing, the use of homogeneous coordinates generally leads to simpler formulas that involve only the basic operations of linear algebra: determinants, dot and cross products, matrix multiplications, and the like. All Euclidean and affine transformations, and all perspective projections, can be expressed as linear maps acting on the homogeneous coordinates of points. For example, the Cartesian coordinates of the point where the lines $ax + by + c = 0$ and $rx + sy + t = 0$ intersect are

$$\frac{(bt - cs,\ cr - at)}{as - br}.$$

In homogeneous coordinates, the intersection of $\langle a, b, c\rangle$ and $\langle r, s, t\rangle$ is

$$[\, bt - cs,\ cr - at,\ as - br \,]$$

which is easily recognized as the cross product of the vectors (a, b, c) and (r, s, t). As this example shows, with homogeneous coordinates we can eliminate most of the division steps in geometric formulas; the savings are usually enough to offset the cost of handling an extra coordinate. The absence of division steps also makes it possible to do exact geometric computations with all-integer arithmetic.

2.2. Less special cases

Homogeneous coordinates let us handle points and lines at infinity in a natural way, without *ad hoc* flags and conditional statements. Such objects are valid inputs in many geometric applications, and are generally useful as "sentinels" in algorithms (in sorting, merging, list traversal, and so forth). They also allow us to reduce the number of special cases in theorems and computations. For example, when computing the intersection of two lines we don't have to check whether they are parallel. The general line intersection formula will work even in this case, producing a point at infinity. This point can be used in further computations as if it were any ordinary point. By contrast, in the Euclidean or Cartesian models we must disallow this special case, or explicitly test for it and handle it separately. Note that when we compose two procedures or theorems, their special cases usually get multiplied rather than added. Therefore, even a small reduction in the special cases of basic operations — say, from three to two — will enormously simplify many geometric algorithms.

2.3. Unification and extension of concepts

Another advantage of projective geometry is its ability to unify seemingly disparate concepts. For example, the differences between circles, ellipses, parabolas, and hyperbolas all but disappear in projective geometry, where they become instances of the same curve, the non-degenerate conic.

When several disparate concepts are unified into a more general idea, the latter often turns out to include several interesting special cases that were not covered by the former. This happens in projective geometry, too. For instance, all Euclidean and affine transformations (translations, rotations, similarities, and so on) are unified in the idea of *projective map*, a function of points to points and lines to lines that preserves incidence. Besides those familiar transformations, this class contains many new and interesting ones, such as the perspective maps. In Euclidean geometry these maps cannot even be properly defined, since they exchange some finite points with infinite ones.

2.4. Duality

Duality is another powerful tool that is available only in projective geometry. Consider the one-to-one function '$*$' that associates the point $[w, x, y]$ to the line line $\langle w, x, y \rangle$, and vice-versa. This mapping preserves incidence: if point p is on line l, then line p^* passes through point l^*. The existence of such a map ultimately implies that every definition, theorem, or algorithm of projective geometry has a *dual*, obtained by exchanging the word "point" with the word "line," and any previously defined concepts by their duals. For example, the assertion "there is a unique line incident to any two distinct points" dualizes to "there is a unique point incident to any two distinct lines."

Duality is extremely useful in theory and practice; thanks to it, every proof automatically establishes the correctness of two very different theorems, and every geometrical algorithm automatically solves two very different problems. In Cartesian geometry we can get such a duality only at the cost of leaving out certain lines (e.g., the vertical ones, or those passing through the origin). This leads to unnecessarily complicated theorems, and to algorithms with lots of special cases.[6,8]

3. Drawbacks of classical projective geometry

In spite of its advantages, the projective plane has a few peculiar features that are rather annoying from the viewpoint of computational geometry. Some of those problems, which were described in detail by Riesenfeld[9], are:

- *The projective plane is not orientable.* Informally, this means there is no way of defining "clockwise" or "counterclockwise" turns that is consistent over the whole plane $\mathbf{P}_2$. The reason is that a turn can be continuously transported over the projective plane in such a way that it comes back to its original position but with its sense reversed. For the same reason, it is impossible to tell whether two triangles (ordered triplets of points) have the same or opposite handedness. This is quite inconvenient, since these two tests are the building blocks of many geometric algorithms.

- *Lines have only one side.* If we remove a straight line from the projective plane, what remains is a *single* connected set of points, topologically equivalent to a disk. Therefore, we cannot meaningfully ask whether two points are on the same side of a given line. More generally, *Jordan's theorem is not true* in the projective plane, since a simple closed curve (of which a straight line is a special case) need not divide the plane in two distinct regions. Even if we consider only the immediate neighborhood of a line, we still cannot distinguish its two sides,

since that neighborhood has the topology of a Möbius band. See figure 5.

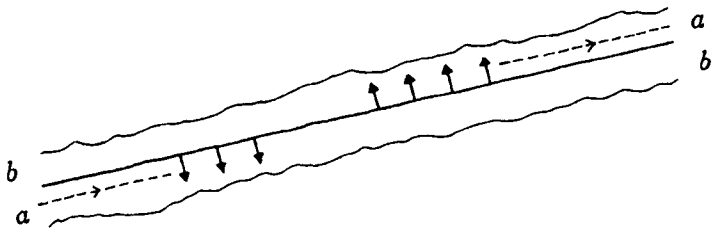

Figure 5. The neighborhood of a straight line of $\mathbf{P}_2$.

- *Segments are ambiguous.* In projective geometry we cannot define *the* line segment connecting two points in a consistent way. Two points divide the line passing through them in *two* simple arcs, and there is no consistent way to distinguish one from the other. It is therefore impossible to tell whether a point r lies between two given points p, q.

- *Directions are ambiguous.* By the same token, we cannot define *the* direction from point p to point q. In particular, each point at infinity lies simultaneously in two opposite directions, as seen from a finite point. This property often makes it hard to use points at infinity as "sentinels" in geometric algorithms and data structures.

- *There are no convex figures.* The notion of convex set has no meaning in projective geometry. The problem is not just that the classical definition of convex set ("one that contains every segment joining two of its points") becomes meaningless, but in fact that there is no consistent way to distinguish between convex and non-convex sets.

Of course, we can avoid all these problems by letting our definitions of segment, direction, and so on depend on a special line Ω. However, we would then have to exclude certain "degenerate" cases, such as segments with endpoints on Ω. The concepts thus defined will not be preserved by arbitrary projective maps and will have uninteresting duals. In fact, this "solution" means giving up projective geometry, and retreating to the Euclidean world.

4. Oriented projective geometry

Oriented projective geometry retains most advantages of the classical theory, but avoids the problems listed in the previous section. Its primitive objects are points and *oriented flats*: oriented lines, oriented planes, and so on. In particular, every straight line has an intrinsic orientation, which determines a "forward" direction along the line at every one of its points.

Every line of the classical projective plane is thus replaced by two coincident but oppositely oriented (hence distinct) lines. In order to maintain the exact duality between points and lines, each point must also be replaced by two "oppositely oriented" copies. Algebraically, this means treating $[w, x, y]$, $[-w, -x, -y]$ as distinct points, and $\langle W, X, Y\rangle$, $\langle -W, -X, -Y\rangle$ as distinct lines. The resulting set of points is topologically a double covering of the projective plane. Accordingly, I will use *two-sided* as a synonym of *oriented projective*. The set of points is in fact topologically equivalent to a sphere, with straight lines corresponding to oriented great circles. Therefore, oriented projective geometry is simply an oriented version of spherical geometry.

The double covering makes it possible to postulate an intrinsic *circular orientation* for the whole plane, which defines the "positive sense of turning" at every point, in a consistent way. This allows us to talk about the orientation of other objects in absolute terms: we can say that a triangle is positively oriented, without having to specify a "reference" triangle every time. The global orientation of the plane also makes it possible to use the "forward" direction of a line to define its "left" and "right" sides.

In oriented projective geometry, a pair of points p, q generally determines not one but *two* distinct lines, with the same position but opposite orientations. We still can unambiguously speak of *the* line through p to q, if we pay attention to the order of those two points. That is, we must distinguish between the line joining p to q and the one joining q to p. Dually, two lines l and m on the plane have generally two points in common, so we must distinguish the point where l meets m from the point where m meets l. We will see that the two can be unambiguously defined by taking into account the orientations of l and m, and the global orientation of the whole plane.

The previously mentioned advantages of projective geometry are retained in the oriented version. In particular, we are still able to define an exact duality between points and lines that preserves not only the incidence properties of all objects, but also their relative orientations. In addition, the oriented version allows us to define the concept of convexity in a truly projective way. Unlike the Cartesian definition, the new one is unaffected by arbitrary projective maps and duality: we can finally say that the problem of intersecting n half-planes is *exactly* dual to finding the

convex hull of n points, and not *approximately* so. Indeed, the ability to support both convexity and duality is perhaps the greatest advantage of the new framework.

All of this extends quite nicely to higher-dimensional spaces. A major difficulty there is that our geometric intuition becomes less powerful and less reliable. Once we leave the plane, the orientation of objects becomes much harder to visualize and to reason about. A great advantage of oriented projective geometry is that it gives us effective and reliable tools for doing this. Oriented projective geometry can be viewed as the marriage of projective geometry with an *algebra of orientations.*

5. Relation to previous work

Projective geometry and its history are too well documented for me to give them more than a fleeting treatment here. The interested reader should consult any basic textbook on the subject[4] and follow the leads from there.

In the realm of computer graphics, the idea of distinguishing homogeneous tuples that differ by a global sign change is not entirely new. For example, programmers usually compute the sign of $Ww + Xx + Yy$ in order to test whether the point $[w, x, y]$ is on the left side of the line $\langle W, X, Y \rangle$. This obviously distinguishes between that line and the line $\langle -W, -X, -Y \rangle$. However, this distinction is applied unsymmetrically: only to line coefficients, and not to point coordinates. In fact, users of this formula have to ensure that the coordinates of the point $[w, x, y]$ are "sign-normalized" in some consistent way (for example, so that $w > 0$), thus abolishing the distinction between that point and $[-w, -x, -y]$. Besides requiring explicit tests and sign-reversals, this assumption destroys the point-line duality, and is mathematically inconsistent in many other ways (especially in the treatment of points at infinity). For more details, see the Riesenfeld's paper.[9]

The distinction between homogeneous tuples of opposite sign is also commonly made in the perspective rendering of three-dimensional models. One step in this process is applying to the whole three-space a projective transformation which keeps the projection plane fixed and moves the observer to infinity. (Algebraically, this transformation consists of multiplying the homogeneous coordinates of every point by a 4×4 matrix.) This maps all rays out of the observer's eye into parallel lines, so that the perspective projection is reduced to a simpler parallel one. However, it also has the unwanted effect of "folding" those parts of the image that originally were behind the observer (and hence invisible to him) over the visible part of the image.[10] See figure 6.

It turns out that if the homogeneous coordinates of the original points are sign-normalized to have positive weight, then the invisible points (and only those) will have negative weight after the transformation. Although this fact is well-known and

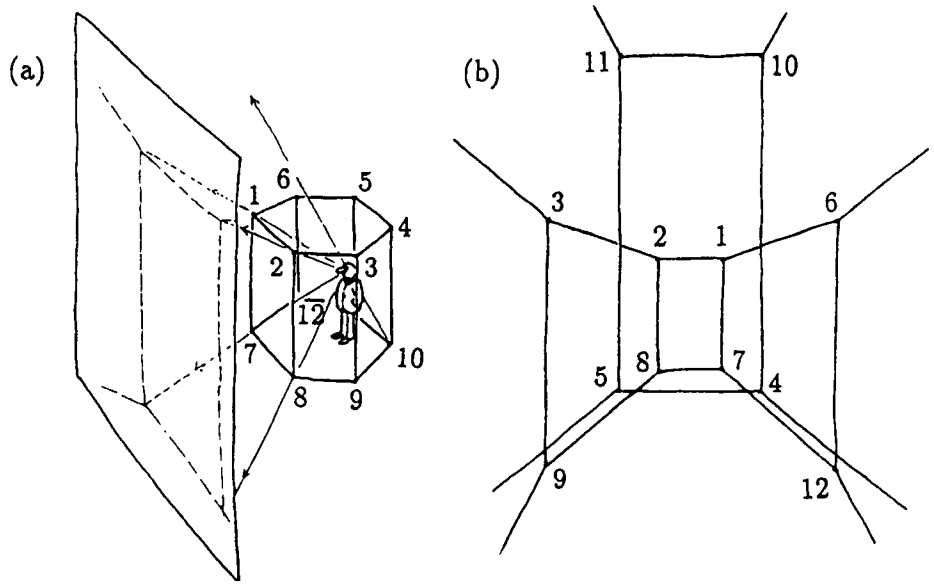

Figure 6.

widely used by graphics programmers, it cannot be explained within the classical theory of homogeneous coordinates. According to that theory, the procedure that does the perspective transformation would be allowed to arbitrarily reverse the sign of all coordinates of the result. The negative-weight clipping rule above is therefore presented as a programming trick, based on implicit assumptions about the inner workings of the transformation procedure.

On the theoretical front, Hermann Grassmann seems to have been the first to consider a geometric calculus based on two dual products (what we call join and meet), about a hundred years ago. His ideas were explored and reformulated by several other mathematicians since then, notably Clifford, Schröder, Whitehead, Cartan, and Peano. For a recent exposition of the ideas involved, see for example the paper by Berman[11] or the book by Hestenes and Sobczyk.[5] For some reason, the geometric calculus developed by those authors was relegated to relative obscurity, and its usefulness for practical computations has been largely ignored so far. Part of the reason may be the highly abstract language, excessive generality, and heavy mathematical formalism used in most expositions, which make the fundamental ideas seem much more complicated than what they really are.

The notation used in this paper is quite similar to the one used in a recent paper by Barnabei, Brini, and Rota,[2] although it was developed independently from their work. The notion of an oriented flat as defined in the next chapters is closely related to what they call an *extensor*, or decomposable antisymmetric tensor. More precisely, the flats of oriented projective geometry are the equivalence classes we obtain by considering two extensors equivalent iff they differ by a positive scalar

factor. Compared to their paper, this thesis gives more emphasis to the geometric (as opposed to algebraic) aspects of the calculus, and in particular to its suitability as the common language of computational geometry.

6. References

[1] C. F. Adler: **Modern geometry** (2nd. ed.). McGraw-Hill (1970).

[2] M. Barnabei, A. Brini, and G.-C. Rota: **On the exterior calculus of invariant theory.** Journal of Algebra vol. 96 no. 1 (Sept. 1985), pp. 120–160.

[3] L. M. Blumenthal and K. Menger: **Studies in geometry** W. H. Freeman (1970).

[4] H. S. M. Coxeter: **The real projective plane.** McGraw-Hill (1949).

[5] D. Hestenes and G. Sobczyk: **Clifford algebra to geometric calculus: A unified language for mathematics and physics.** D. Reidel, Dordrecht (1984).

[6] A. Inselberg and M. Reif: **Convexity in parallel coordinates.** [Manuscript] CS Department, Univ. of California at Los Angeles (June 1984).

[7] P. S. Modenov and A. S. Parkhomenko: **Geometric transformations. vol. II: Projective transformations.** Academic Press (1965).

[8] F. P. Preparata and D. E. Muller: **Finding the intersection of n half-spaces in time $O(n \log n)$.** Theoretical Computer Science vol. 8 (1979), pp. 44–55

[9] R. F. Riesenfeld: **Homogeneous coordinates and projective planes in computer graphics.** IEEE Computer Graphics and Automation (Jan. 1981), pp. 50–55.

[10] A. Penna and R. Patterson: **Projective geometry and its applications to computer graphics.** Prentice-Hall (1986).

[11] G. Berman: **The wedge product.** American Mathematical Monthly vol. 68 no. 2 (1961) 112–119.

Chapter 2
Oriented projective spaces

Rather than defining oriented projective geometry by a list of axioms, I will construct a *canonical two-sided space* $\mathbf{T}_\nu$ for each dimension ν. This object will consist of an oriented manifold Υ_ν (whose elements are called *points*) and a collection $\mathcal{F}_\nu$ of oriented submanifolds of Υ_ν (the *flats*). I will then be able to define a generic oriented projective space as any pair (U, F) isomorphic to $(\Upsilon_\nu, \mathcal{F}_\nu)$ for some ν. (I will generally omit the subscript ν when it is clear from the context.)

1. Models

Actually, I will construct three equivalent versions of $\mathbf{T}_\nu$, analogous to the straight, spherical, and analytic models of $\mathbf{P}_2$:

1.1. The spherical model

The *spherical model* of $\mathbf{T}_\nu$ consists of the unit sphere $\mathbf{S}_\nu$ of $\mathbf{R}^{\nu+1}$, that is, the set of all points $(x_0, .. x_\nu)$ of $\mathbf{R}^{\nu+1}$ such that $\sum {x_i}^2 = 1$. Note that diametrally opposite points are *not* identified. For example, $\mathbf{T}_1$ (the *two-sided line*) is modeled by the unit circle of $\mathbf{R}^2$, and $\mathbf{T}_2$ (the *two-sided plane*) is modeled by the unit sphere of $\mathbf{R}^3$. See figure 1.

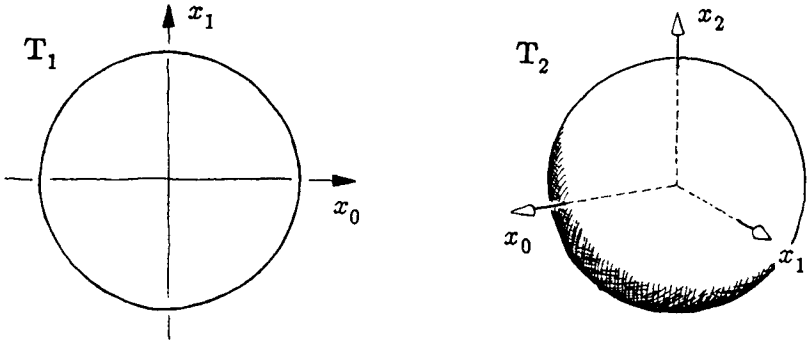

Figure 1. The spherical models of $\mathbf{T}_1$ and $\mathbf{T}_2$.

1.2. The straight model

The *straight model* of $\mathbf{T}_\nu$ consists of two copies of $\mathbf{R}^\nu$ (the *front* and *back ranges*), and one *point at infinity* $d\infty$ for every direction vector d in $\mathbf{R}^\nu$. (In this chapter, *direction* means a unit-length vector.)

For instance, the straight model of $\mathbf{T}_1$ consists of two copies of the real line $\mathbf{R}$, and two points at infinity $+\infty$ and $-\infty$. We can visualize this model as an infinite ruler with graduated scales on both sides. See figure 2.

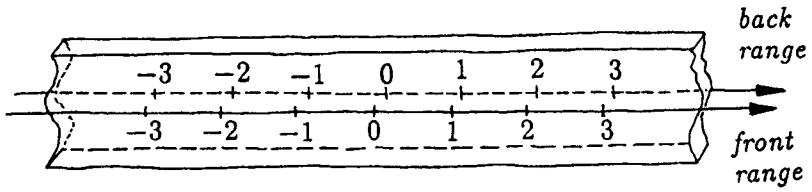

Figure 2. The straight model of $\mathbf{T}_1$.

Similarly, the straight model of $\mathbf{T}_2$ consists of two copies of $\mathbf{R}^2$, and an infinity point $d\infty$ for every direction d of $\mathbf{R}^2$. We can visualize the front and back ranges as two parallel planes in three space, or as the two sides of an infinite sheet of paper. Figure 3 is a sketch of this model, where the front and back ranges are represented by two copies of the open unit disk. The infinity point $d\infty$ is represented by point d on the boundary of the front disk, and point $-d$ on the boundary of the other.

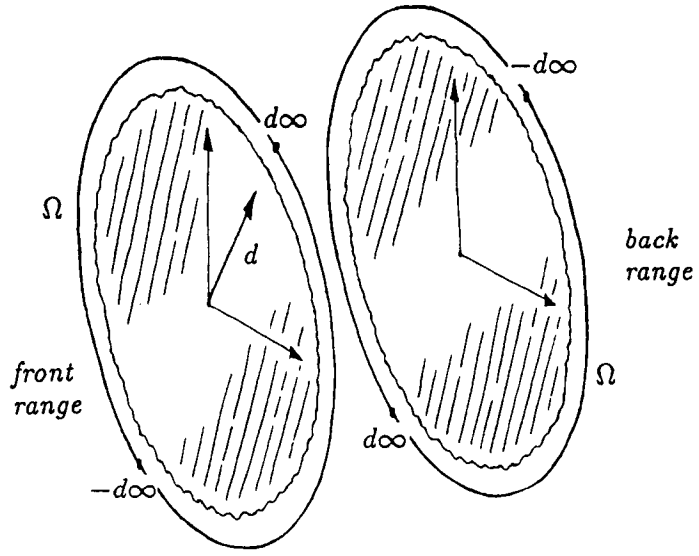

Figure 3. The straight model of $\mathbf{T}_2$.

This contracted version gives a more accurate picture of the topology of $\mathbf{T}_2$, in particular around the line at infinity. Note that the infinity points $d\infty$ and $(-d)\infty$ are *not* identified (unlike the conventions of standard projective geometry). Each infinity point is incident to both ranges, but in a rather peculiar way: by definition, $d\infty$ is a limit point of the front range in the direction d, and of the back range in the *opposite* direction $-d$.

The straight model suggests a convenient representation for figures on the two-sided plane: simply draw the front and back parts on the same sheet of paper, with coincident coordinate frames, using different graphical styles for each range. I will use solid dots, solid lines and cross-hatching for elements on the front range, and open dots, dashed lines, and dotted patterns for the back range. See figure 4.

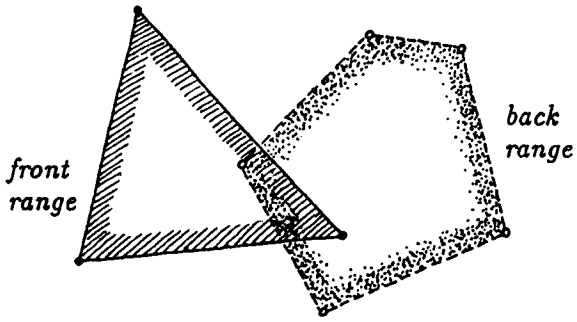

Figure 4. Graphical conventions for the two-sided plane.

Figure 5 is a sketch of the straight model of three-dimensional oriented projective space $\mathbf{T}_3$, with each range contracted down to a copy of the unit open ball of $\mathbf{R}^3$. A point at infinity $d\infty$ is represented by point d on the boundary of the first ball, and point $-d$ on the boundary of the second.

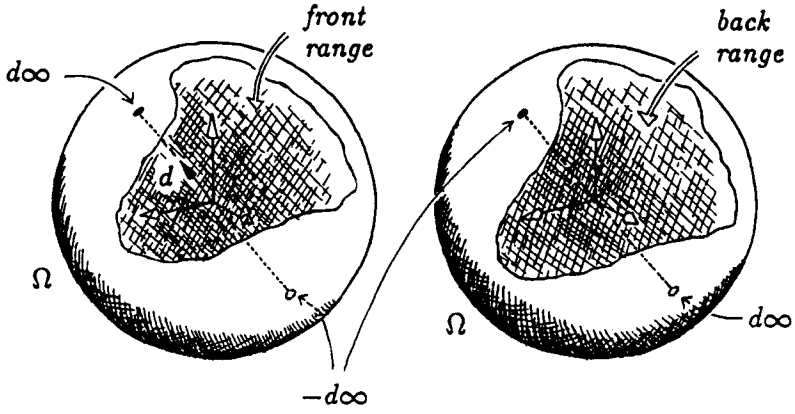

Figure 5. A sketch of the straight model of $\mathbf{T}_3$.

We can imagine the front half as being the set of "real" points, and the back half as a parallel universe of "phantom" points. Both ranges extend throughout the whole $\mathbf{R}^3$, but the only way to go from one to the other is through the points at infinity.

Recall that the *unoriented* projective space $\mathbf{P}_\nu$ can be constructed from $\mathbf{R}^\nu$, by adding to it one point at infinity for every pair of opposite directions $\{+d, -d\}$. The straight model of $\mathbf{T}_\nu$ is clearly a double covering of this construction.

1.3. The analytic model

The *analytic model* of $\mathbf{T}_\nu$ consists of the non-zero vectors of $\mathbf{R}^{\nu+1}$, where two vectors are considered to be the same point if one is a *positive* multiple of the other. I will denote by $[u]$ or $[u_0, .. u_\nu]$ the point represented by the vector $u = (u_0, .. u_\nu)$ and its positive multiples; any of those vectors is called the *(signed) homogeneous coordinates* of that point. Obviously, $[u_0, .. u_\nu] = [v_0, .. v_\nu]$ if and only if $u_i = \alpha v_i$ for all i and some *positive* real α. Note that $[u_0, .. u_\nu]$ and $[-u_0, .. -u_\nu]$ are distinct points of $\mathbf{T}_\nu$.

2. Central projection

The three models are related by *central projection* from the origin of $\mathbf{R}^{\nu+1}$. A point $[w, x, y, \ldots, z]$ of the analytic model corresponds to the points

$$\frac{(w, x, y, \ldots, z)}{\sqrt{w^2 + x^2 + y^2 + \cdots + z^2}}$$

of the spherical model and $(x/w,\ y/w\ \ldots,\ z/w)$ of the straight model. By definition, the latter is on the front range if $w > 0$, on the back range if $w < 0$, and at infinity in the direction $(x, y, \ldots, z)$ if $w = 0$.

2.1. Central projection of the two-sided line

In the case of $\mathbf{T}_1$, for example, central projection identifies the homogeneous pair $[w, x]$ with the point x/w of the front or back range of $\mathbf{T}_1$, depending on whether $w > 0$ or $w < 0$. The points $(0, 1)$ and $(0, -1)$ are mapped to $+\infty$ and $-\infty$, respectively.

Geometrically, this process can be described as follows. First, we draw the *front* range of $\mathbf{T}_1$ on the plane $\mathbf{R}^2$, as a vertical axis with its origin at the point $(1, 0)$. See figure 6(a). The points on the *left* half of the circle are then projected onto this axis, by straight rays emanating from the origin $(0, 0)$. For the other half of the circle, we let that same vertical axis represent the *back* range of $\mathbf{T}_1$. The points on the *right* half of the unit circle are projected on this axis *across* the center, as shown in figure 6(b).

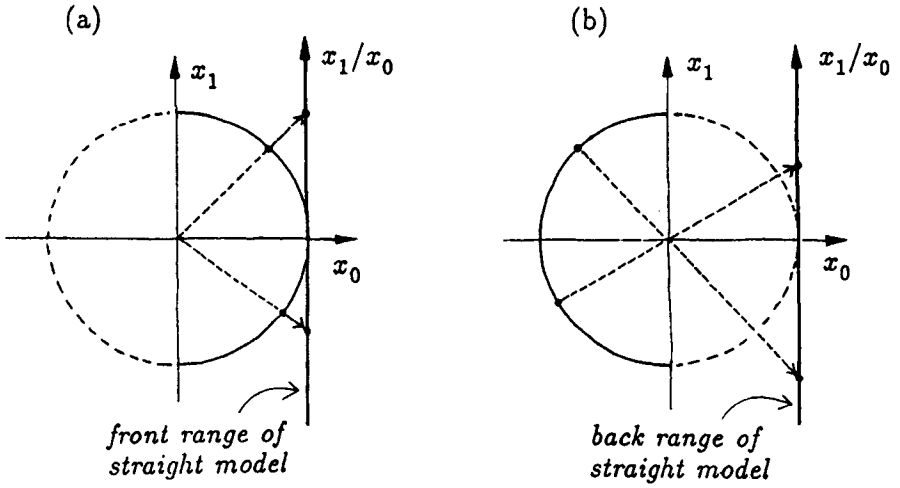

Figure 6. Central projection of $\mathbf{T}_1$.

Algebraically, central projection allows us to view a point $[w, x]$ of $\mathbf{T}_1$ as the fraction x/w, provided we distinguish it from the fraction $(-x)/(-w)$. The two fractions lie on different ranges, but have the same *numerical value*, that is, they have the same position within their range. As we will see in chapter 15, we can operate with these "two-sided fractions" in pretty much the same way we operate with normal ones.

2.2. Central projection of the two-sided plane

The two-dimensional case is entirely analogous. Imagine that the front range of $\mathbf{T}_2$ is embedded in $\mathbf{R}^3$, with the origin at $(1, 0, 0)$ and coordinate axes parallel to $(0, 1, 0)$ and $(0, 0, 1)$. See figure 7(a).

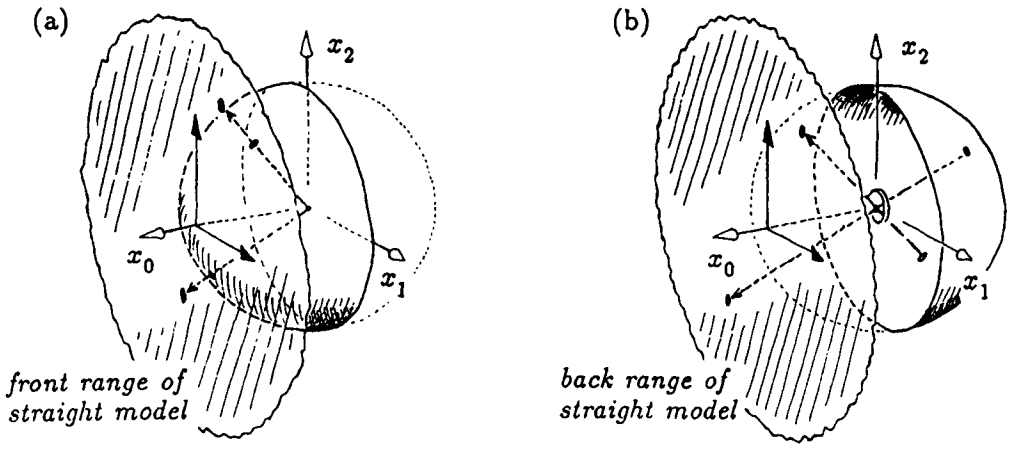

Figure 7. Central projection of $\mathbf{T}_2$.

Now imagine a light source at the center of the unit sphere $\mathbf{S}_2$. Central projection

takes every point of the "front" hemisphere of $\mathbf{S}_2$ (i.e., with $w > 0$) to the shadow it casts on the front face. For the other half of the mapping, we must place the back range of $\mathbf{T}_2$ in exactly the *same* position and orientation. Then points on the back hemisphere of $\mathbf{S}_2$ are projected onto it across the point $(0,0,0)$, as if through a camera lens. See figure 7(b). Here are some examples:

$$\begin{aligned}
[+3, +2, +6] &\mapsto \left(+\tfrac{2}{3}, +\tfrac{6}{3}\right) && \text{front}\\
[-3, -2, -6] &\mapsto \left(+\tfrac{2}{3}, +\tfrac{6}{3}\right) && \text{back}\\
[+3, -2, -6] &\mapsto \left(-\tfrac{2}{3}, -\tfrac{6}{3}\right) && \text{front}\\
[-3, +2, +6] &\mapsto \left(-\tfrac{2}{3}, -\tfrac{6}{3}\right) && \text{back}\\
[\ 0, +3, +4] &\mapsto (+3, +4)\infty && \text{infinity}\\
[\ 0, -3, -4] &\mapsto (-3, -4)\infty && \text{infinity}
\end{aligned}$$

Observe how the front hemisphere is merely stretched and flattened out by this projection, whereas the back hemisphere suffers an additional 180° rotation.

2.3. Final comments

I will adopt central projection as the standard correspondence between the three models, and generally think of them as the same mathematical object. In definitions and theorems I will use whichever model is more convenient, and let central projection implicitly carry the same concepts to the other two.

The analytic model is the most convenient to use in actual computations and data structures. The other two are useful mainly as visual aids in the interpretation of problems and the derivation of algorithms. The straight model is of course the link between Euclidean and oriented projective geometry, since the front range of $\mathbf{T}_\nu$ is a faithful model of Euclidean space. The spherical model of $\mathbf{T}_1$ and $\mathbf{T}_2$ makes it easier to visualize their topological and geometric properties (particularly at the infinity points). Unfortunately, the spherical model of $\mathbf{T}_\nu$ isn't nearly as useful for $\nu \geq 3$, since the geometry of $\mathbf{S}_\nu$ is then hard to visualize.

Chapter 3
Flats

The geometric structure of $\mathbf{T}_\nu$ is largely determined by its *flats*. These are sets of points roughly equivalent to the lines, planes, and higher-dimensional subspaces of classical geometry. One major difference is that every flat of $\mathbf{T}_\nu$ has an intrinsic *orientation*: a 1-dimensional flat is like a *directed* line, a 2-dimensional flat is like a plane with a built-in notion of "positive turn," and so on.

Since most geometric computations are confined to the plane or to three-dimensional space, a good part of this chapter will be devoted to an informal description of the two-sided spaces $\mathbf{T}_2$ and $\mathbf{T}_3$.

1. General definitions

Definition 1. In the spherical model of $\mathbf{T}_\nu$, a *flat set* is a great sphere of $\mathbf{S}_\nu$, that is, the intersection of $\mathbf{S}_\nu$ and some linear subspace of $\mathbf{R}^{\nu+1}$. A *flat* is an oriented flat set, that is, an oriented great sphere of $\mathbf{S}_\nu$.

A precise definition of "oriented" will be given in the next chapter; for now, it suffices to say that a great sphere can be oriented in exactly two ways. So, for every flat a there is an *opposite* flat $\neg a$, consisting of the same set of points (the same great sphere) taken with opposite orientation. Needless to say, I will always regard a and $\neg a$ as distinct flats.

1.1. Special cases

Flats of dimension 1,2, and 3 are called *lines*, *planes*, and *three-spaces*. There are only two flats with dimension ν, namely the universe Υ_ν and its oppositely oriented version $\neg\Upsilon_\nu$. Flats with dimension $\nu - 1$ are called *hyperplanes*. Note that a κ-dimensional flat of $\mathbf{T}_\nu$ is essentially a copy of the universe Υ_κ of $\mathbf{T}_\kappa$; this observation will be made more precise later on.

A flat set of dimension zero consists of two antipodal points of $\mathbf{S}_\nu$. Orienting such a set is equivalent (as we will see) to picking one of the two points as the "positive" one. Therefore, the zero-dimensional (oriented) flats can be identified with the points of $\mathbf{T}_\nu$.

If p is a point, its opposite $\neg p$ is also called its *antipode*; it is the point diametrally opposite to p in the spherical model. See figure 1(a). Observe that such

antipodal pairs are *not* identified.

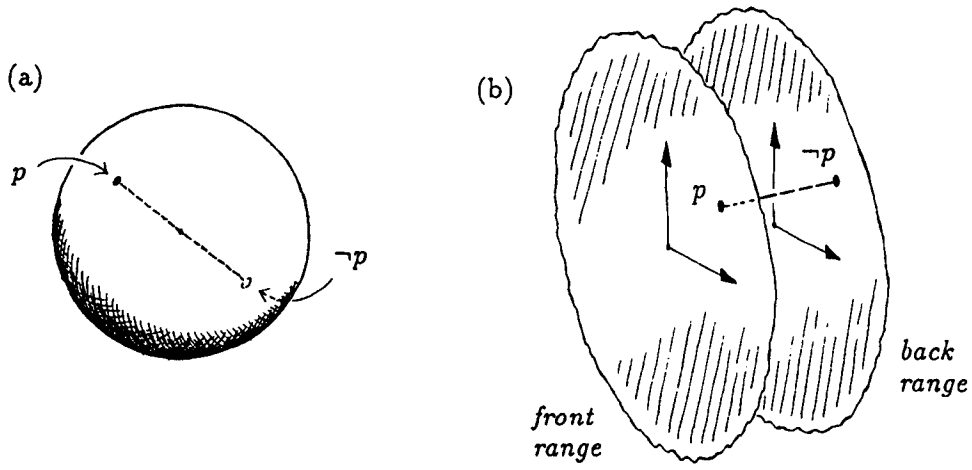

Figure 1. Antipodal points of $\mathbf{T}_2$ in the spherical and straight model.

If we follow two antipodal points through central projection, we will see that in the straight model they (1) are both points at infinity, in diametrally opposite directions, or (2) have the same Cartesian coordinates, but lie on opposite ranges of $\mathbf{T}_\nu$. See figure 1(b). In the analytic model, the antipode of $[w, x, y, .. z]$ is $[-w, -x, -y, .. -z]$. It is convenient to define $\sigma \circ f$, for any flat f and any $\sigma \in \{\pm 1\}$, as being f if $\sigma = +1$, and $\neg f$ if $\sigma = -1$.

In addition to the flats defined above, I will postulate two flats of rank 0 and dimension -1, the *positive vacuum* Λ and its opposite, the *negative vacuum* $\neg\Lambda$. They should be regarded as oriented versions of the empty set.

1.2. Ranks

In projective geometry, it is often convenient to classify spaces and flats by their *rank*, defined to be their dimension plus one. Thus, for example, points have rank 1, lines have rank 2, planes have rank 3, and so on. Ranks, as opposed to dimensions, seem to arise most frequently in formulas.

To keep formulas short and reduce the possibility of confusion, I will adopt the following convention: the greek letters $\kappa, \mu, \nu, \rho, \sigma, \tau$ will usually denote dimensions, and the corresponding italic letters k, m, n, r, s, t will usually denote the corresponding ranks. The identities $k = \kappa + 1$, $m = \mu + 1$, and so on will be assumed throughout. With this convention we can say, for example, that $\mathbf{S}_\nu$ is the unit sphere of $\mathbf{R}^n$, and a κ-dimensional great subsphere of $\mathbf{S}_\nu$ is the intersection of $\mathbf{S}_\nu$ and a k-dimensional linear subspace of $\mathbf{R}^n$.

I will denote the set of all flats of rank k in $\mathbf{T}_\nu$ by $\mathcal{F}^k_\nu$, or simply $\mathcal{F}^k$ when ν is clear from the context.

2. Lines

In the spherical model, a line of $\mathbf{T}_\nu$ is an oriented great circle of the unit sphere $\mathbf{S}_\nu$. The orientation can be visualized as an arrow that tells which direction along the circle is *positive* ("forward"). The opposite $\neg l$ of a line l is the same great circle with the arrow going the other way. See figure 2.

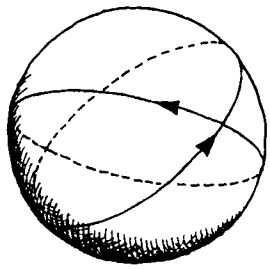

Figure 2. Lines of $\mathbf{T}_2$.

The one-dimensional space $\mathbf{T}_1$ has exactly two lines, namely $\mathbf{S}_1$ itself in its two possible orientations. By definition, the universe Υ_1 of $\mathbf{T}_1$ is oriented counter-clockwise, that is, from $(1,0)$ to $(0,1)$ by the shortest route. See figure 3.

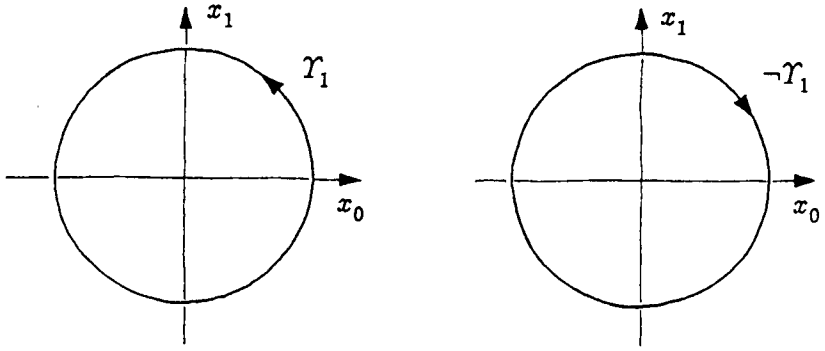

Figure 3. The two lines of $\mathbf{T}_1$.

2.1. Lines in the straight model

In order to understand what a line looks like in the straight model, let us begin with the one-dimensional case, namely the universe of $\mathbf{T}_1$. Imagine a point p moving counterclockwise on $\mathbf{S}_1$, and consider its image on the straight model under central projection. See figure 4.

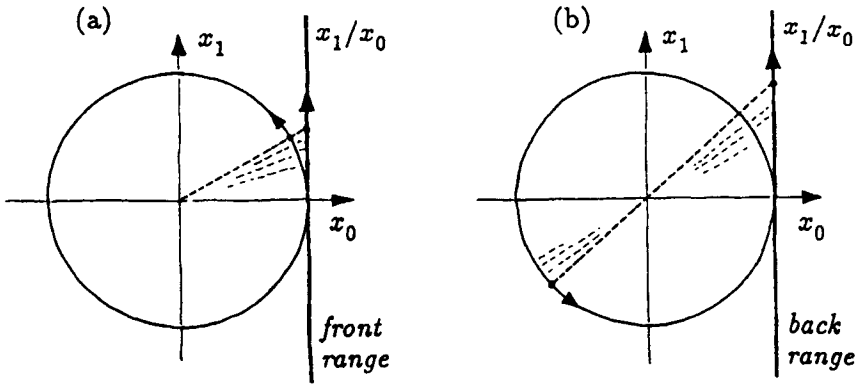

Figure 4.

While p is on the right half of $\mathbf{S}_1$, its image scans all points on the front range, in increasing order. As p goes through the point $(0,1)$ at the top of the circle, its image becomes the infinity point $+\infty$, and then suddenly jumps to the back range, at large *negative* values. It then traverses the entire back range, *again in increasing order*. When p goes through the point $(0,-1)$, its image jumps to the other infinity point $-\infty$, and then to the front range, again at the *negative* end. Therefore, a cyclic ordering of the points of $\mathbf{S}_1$ corresponds in the straight model to either increasing or decreasing order of points on the front range, and the *same* ordering on the back range. Figure 5 shows the two lines of $\mathbf{T}_1$.

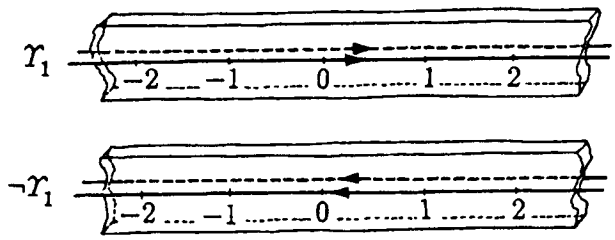

Figure 5. The lines of $\mathbf{T}_1$ in the straight model.

Let's now consider lines in higher-dimensional spaces. We can verify that central projection of a great circle of $\mathbf{T}_2$ gives either (1) an *improper* line, consisting

of an oriented circle of points at infinity, or (2) *a proper* line, consisting of two copies of the same directed Euclidean straight line, one on the front range and one on the back range, plus the two points at infinity incident to them. See figure 6.

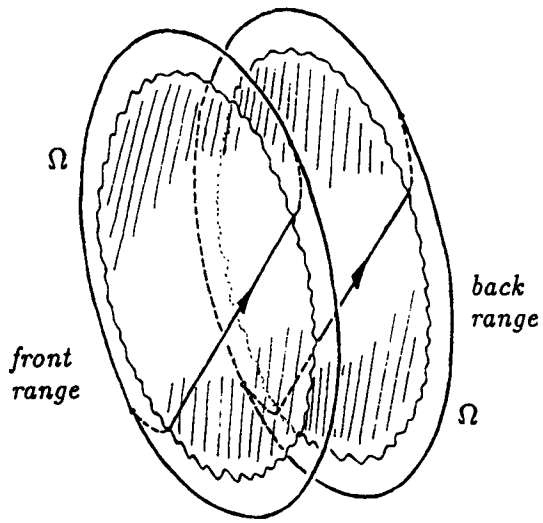

Figure 6. A proper line in the straight model of $\mathbf{T}_2$.

In the two-sided plane there are only two improper lines; by definition, the *horizon* Ω_2 of $\mathbf{T}_2$ is the one oriented counterclockwise (as seen from the front range). In general, an improper line of $\mathbf{T}_\nu$ consists of all points at infinity whose directions lie on some plane through the origin of $\mathbf{R}^\nu$.

Note that the front and back parts of a proper line are directed the *same* way. That is, the line goes continuously in some direction d, from minus infinity to plus infinity across the front, again from minus infinity to plus infinity, in the same direction across the back.

This apparent "jump" when a line crosses the horizon may seem somewhat disconcerting. We could avoid it by redefining the straight model with the back range rotated 180 degrees, but this would make other important concepts (such as antipodal pairs, half-planes, and convex figures) much harder to visualize. Moreover, it would introduce explicit sign tests and conditional sign changes in the conversion of homogeneous to Cartesian coordinates.

Note how proper and improper lines look very much like the straight and spherical models of $\mathbf{T}_1$, respectively. This is true in general: in the straight model of $\mathbf{T}_\nu$, a κ-dimensional flat either looks like the spherical model of $\mathbf{T}_\kappa$, expanded to infinite radius, or looks like the straight model of $\mathbf{T}_\kappa$, embedded in that of $\mathbf{T}_\nu$ in the obvious way.

3. Planes

A plane (two-dimensional flat) is an oriented great 2-sphere of $\mathbf{S}_\nu$. The orientation can be visualized as a small "circular arrow" painted on the sphere. By sliding that arrow around the surface, we can tell whether a turn at any point is *positive* (agreeing with the arrow) or *negative*. The opposite $\neg\pi$ of a plane π is the same 2-sphere with the circular arrow turning the other way.

By definition, the universe Υ_2 of $\mathbf{T}_2$ has the circular arrow at $(1,0,0)$ turning from direction $(0,1,0)$ to direction $(0,0,1)$ by the shortest angle. If the axes of $\mathbf{R}^3$ are arranged in space in the usual way, then the standard orientation corresponds to a counterclockwise turn on the sphere, as seen from the outside. See figure 7.

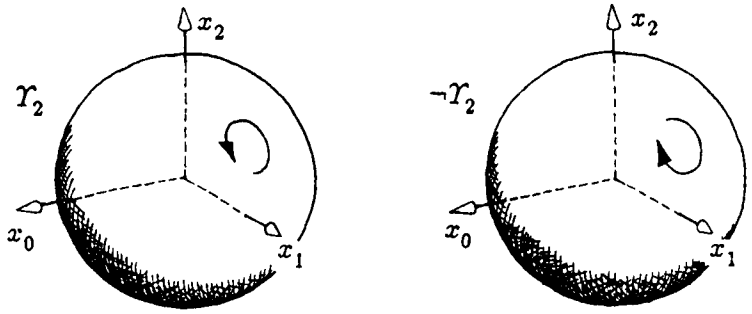

Figure 7. The two planes of $\mathbf{T}_2$.

3.1. Planes in the straight model

What does the orientation of a plane mean in the straight model? If we consider how central projection affects the direction of turns at various points, we see immediately that positive turns of Υ_2 become counterclockwise on the front range, and *clockwise* on the back range. See figure 8.

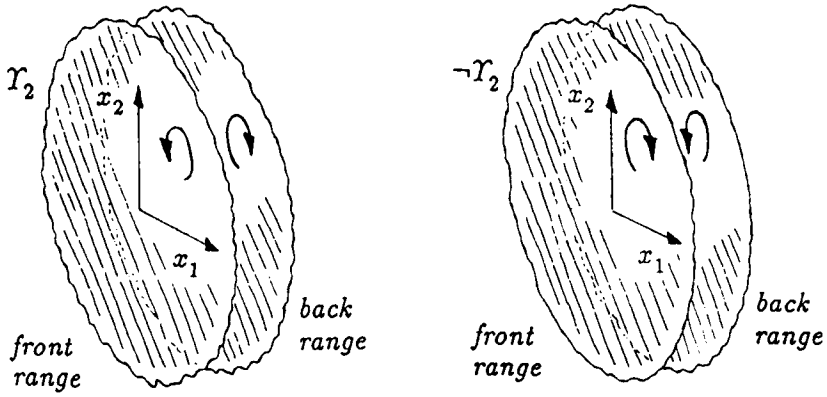

Figure 8. The planes of $\mathbf{T}_2$ in the straight model.

In general, a plane of $\mathbf{T}_\nu$ is either (1) an *improper* plane, consisting of a 2-sphere of points at infinity, or (2) a *proper* plane, consisting of two oppositely oriented copies of the same plane of $\mathbf{R}^\nu$, one on each range, and all points at infinity in directions parallel to that plane. See figure 9.

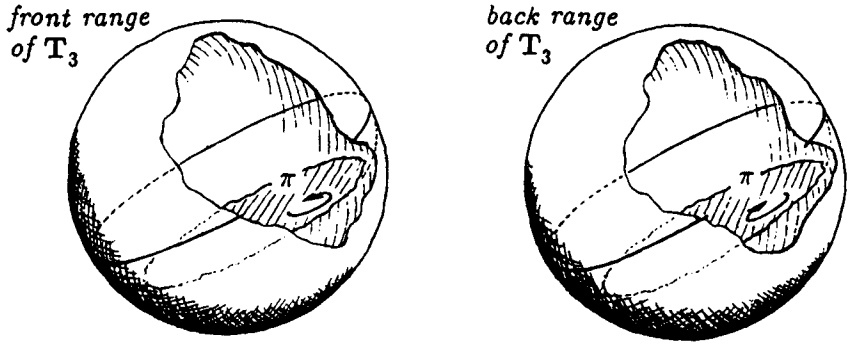

Figure 9. A proper plane of $\mathbf{T}_3$.

In oriented projective three-space there are exactly two oppositely oriented improper planes, consisting of all points at infinity. By definition, the *celestial sphere* Ω_3 is the one oriented *clockwise* as seen from the front range, and counter-clockwise as seen from the back range. See figure 10.

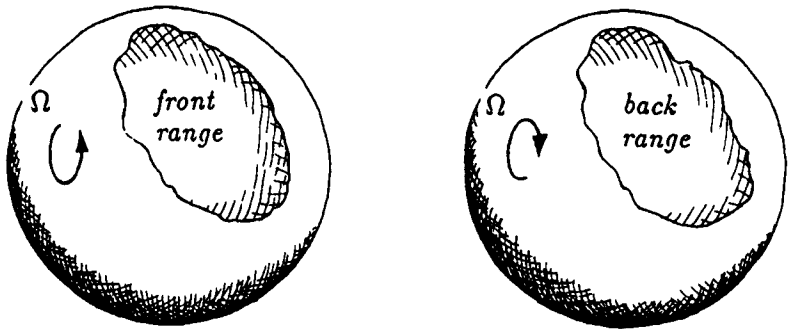

Figure 10. The celestial sphere of $\mathbf{T}_3$.

Observe that if we restrict our attention to the front range of $\mathbf{T}_\nu$ and to the front part of every flat, we get the geometric structure of the ν-dimensional Euclidean space. Therefore, oriented projective geometry is able to simulate all the constructions and algorithms of affine, Euclidean and Cartesian geometry.

4. Three-spaces

A three-space (three-dimensional flat) is an oriented great 3-sphere of $\mathbf{S}_\nu$. We can visualize the orientation of a three-space as a "corkscrew" arrow, or as a combination of a circular arrow and a straight arrow perpendicular to it. We can also depict the orientation as a tiny hand, with the stretched thumb representing the straight arrow and the curled fingers replacing the curved arrow. See figure 11.

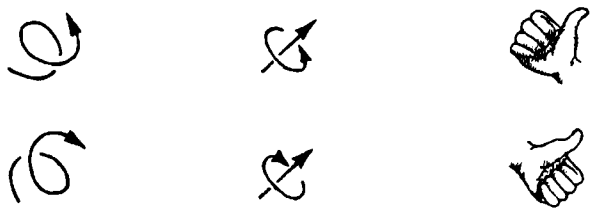

Figure 11. Three-dimensional orientation.

Two such devices represent the same orientation if we can transform one into the other by a continuous rigid motion without leaving the three-space. Therefore, the difference between the two orientations of a three-space correspond to the difference between a left hand and a right hand, or between two oppositely-threaded screws.

Recall that in the straight model a three-space consists of two copies of $\mathbf{R}^3$, plus a sphere of points at infinity. If we consider what happens when we transport a corkscrew arrow across the sphere at infinity by a continuous motion, we will find that picking an orientation for the three-space means picking the *same* orientation on the front and back ranges. In particular, the space $\mathbf{T}_3$ contains only two three-dimensional flats, the standard universe Υ_3 and its opposite $\neg\Upsilon_3$.

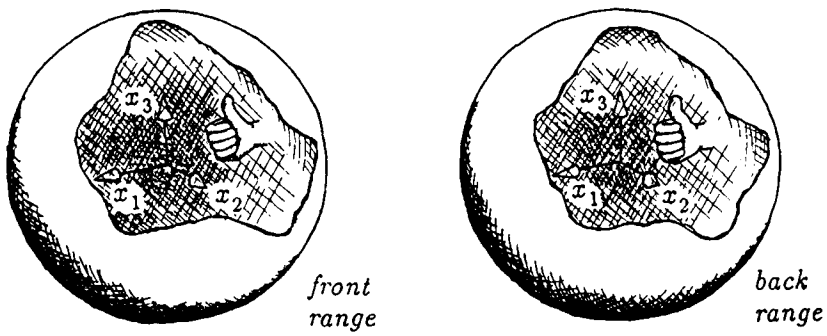

Figure 12. The universe of $\mathbf{T}_3$.

By definition, the orientation of Υ_3 is given by a screw arrow that turns from $(1,0,0)$ towards $(0,1,0)$ while moving from $(0,0,0)$ towards $(0,0,1)$. If the coordinate axes of $\mathbf{R}^3$ are depicted according to the usual mathematical conventions, this orientation is given by thumb and fingers of the *right* hand. See figure 12

It is worth emphasizing that in the straight model, a line or a three-space has both ranges oriented the same way, whereas a plane has the two ranges oriented in opposite ways. The general rule will be given in the following chapter.

Chapter 4
Simplices and orientation

Recall that a κ-dimensional flat of $\mathbf{T}_\nu$ was previously defined as an oriented κ-dimensional great sphere of $\mathbf{S}_\nu$; I will now define more precisely what "oriented" means. It is possible to define "oriented manifold" in purely topological terms, but that requires some heavy mathematical machinery and would take us far from the focus of this thesis. Fortunately, for the particular manifolds we are interested here in (great spheres of $\mathbf{S}_\nu$) we can give a much simpler definition, based on elementary linear algebra.

1. Simplices

A *simplex* is an ordered tuple of points of $\mathbf{T}_\nu$, the *vertices* of the simplex. In the spherical model, a simplex is an ordered tuple of unit vectors of $\mathbf{R}^n$. The simplex is *proper* if those vectors are linearly independent; otherwise it is *improper*, or *degenerate*.

Let us consider some examples. A two-vertex simplex is an ordered pair of points p, q; that simplex is degenerate if and only if the two points are equal or antipodal. See figure 1.

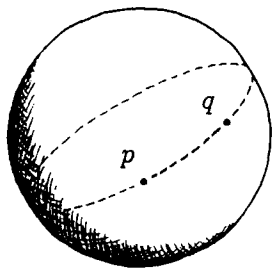

Figure 1. A two-vertex simplex of $\mathbf{T}_2$.

A three-vertex simplex is degenerate if and only if its vertices lie on the same great circle of $\mathbf{S}_\nu$, that is, are coplanar vectors of $\mathbf{R}^n$. This simply means the three points

lie on the same line of $\mathbf{T}_\nu$. See figure 2.

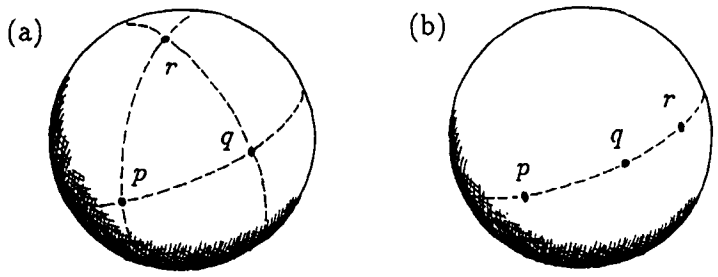

Figure 2. Three-vertex simplices of $\mathbf{T}_2$: (a) proper, (b) degenerate.

For any simplex s of $\mathbf{T}_\nu$ there is a unique flat set of minimum dimension that contains s, the so-called *span* of s. In the spherical model, the span of s is the great sphere of $\mathbf{S}_\nu$ determined by the linear subspace of $\mathbf{R}^n$ generated by the vertices of s (viewed as vectors). The *dimension* of s is by definition that of its span. Obviously, a simplex with k vertices is proper if and only if its dimension is $\kappa = k-1$. Thus, for example, the span of a proper simplex with two vertices is the unique great circle of $\mathbf{S}_\nu$ that contains those two points.

2. Simplex equivalence

I will say that two proper simplices of $\mathbf{T}_\nu$ are *equivalent* if we can continuously transform one into the other in such a way that all intermediate stages span the same unoriented flat. For example, two proper simplices with three vertices are equivalent if they lie on the same unoriented plane of $\mathbf{T}_\nu$, and we can continuously move one onto the other without leaving that plane and without ever making the three points collinear.

2.1. Equivalence of bases

This notion of simplex equivalence is closely related to that of *basis equivalence* in a real vector space V. Two bases of V are said to be equivalent if it is possible to continuously deform one into the other, in such a way that every intermediate stage is a basis of V.

Observe that a proper simplex that spans a great sphere C is a basis for the subspace V of $\mathbf{R}^n$ that defines C. Conversely, from any basis of V we can get a proper simplex spanning C by reducing its vectors to unit length. Since the vectors of a basis have nonzero length, this map from basis to simplex is continuous. We

conclude that two proper simplices are equivalent if and only if they are equivalent bases of the same subspace of $\mathbf{R}^n$.

The condition for two bases to be equivalent is a well-known result of elementary linear algebra. In order to state this condition concisely, it is convenient to view an ordered sequence s of k vectors (in particular, a k-vertex simplex) as a matrix

$$s = \begin{pmatrix} s^0 \\ s^1 \\ \vdots \\ s^\kappa \end{pmatrix}$$

whose rows are the given vectors. (To save space, I will often write such a matrix also as $(s^0; s^1; .. s^\kappa)$, using semicolons instead of commas to denote vertical stacking.) With this convention, we can say that two sequences of k vectors u, v span the same vector space V if and only if there is a $k \times k$ matrix A such that $Au = v$. Also, if u and v are bases of V, the matrix A is unique and has a non-zero determinant; and, furthermore, the bases are equivalent if and only if the determinant is positive.

2.2. Orientations as simplex classes

By the same token, the simplices spanning a given great sphere (flat set) C of $\mathbf{S}_\nu$ are also divided into two equivalence classes. I will identify these two classes with the two *orientations* of C. By naming one of these classes the set of *positive* bases, we get an *oriented great sphere* of $\mathbf{S}_\nu$, i.e. an (oriented) flat of $\mathbf{T}_\nu$. Therefore, a proper κ-dimensional simplex $s = (s^0; .. s^\kappa)$ determines a unique flat of $\mathbf{T}_\nu$, namely the smallest flat set containing s, oriented so that s is a positive simplex. I denote this flat by $[s] = [s^0; .. s^\kappa]$. If $s^i = [s^i_0, .. s^i_\nu]$, then I will write that flat also in matrix form as

$$\begin{bmatrix} s^0_0 & \cdots & \cdots & s^0_\nu \\ \vdots & & & \vdots \\ s^\kappa_0 & \cdots & \cdots & s^\kappa_\nu \end{bmatrix}$$

2.3. Orientation of a point

To make these notions clear, let's have a look at the low-dimensional cases. According to the definition, a zero-dimensional flat set is an unordered pair of antipodal points of the sphere. An *oriented* zero-dimensional flat is one such pair, with one of the points singled out as the "positive" simplex. Obviously, zero-dimensional flats can be identified with the points of $\mathbf{T}_\nu$.

2.4. Orientation of two points on a line

A one-dimensional flat set of $\mathbf{T}_\nu$ is a great circle C of $\mathbf{S}_\nu$. An orientation for this set is a class of equivalent non-degenerate one-dimensional simplices.

Consider for example two proper simplices $(p;q)$ and $(r;s)$ of C. The two simplices are equivalent if and only if we can continuously move p to r and q to s, without leaving C, and without going through a degenerate state. At every instant during this motion the simplex $(p;q)$ determines a unique circular ordering of the points of C, namely the one that goes from p to q by the shortest route; this circular ordering can be depicted as a longitudinal arrow on C. See figure 3.

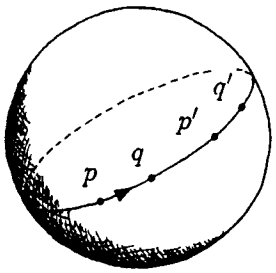

Figure 3.

This ordering remains unchanged while p and q move around, as long as $p \neq q$ and $p \neq \neg q$. It follows that $(p;q)$ and $(r;s)$ are equivalent only if they define the same circular ordering on C. The two orientations of C correspond to its two possible circular orderings. Note that the proper simplex $(p;q)$ is equivalent to $(\neg p;\neg q)$ but not to $(q;p)$ or $(\neg p;q)$.

2.5. Orientation of a triangle

Let's now consider simplices with threee vertices. A proper simplex $(p;q;r)$ defines a unique two-dimensional great sphere C, and a unique spherical triangle on C. See figure 4.

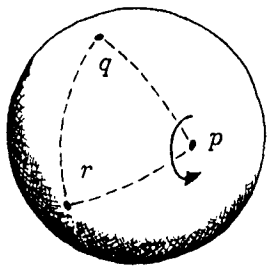

Figure 4. Orientation of simplex $(p;q;r)$.

The sides of the triangle are the shortest arcs of great circle connecting each pair of vertices. Note that for the simplex to be proper no two vertices may be coincident or antipodal.

We can visualize the orientation of the simplex as a small circular arrow surrounding the point p, turning from the direction of pq to that of pr by the shortest angle. Note that the angle between the arcs pq and pr at p cannot be zero or 180°, since in that case the three vertices would be coplanar vectors of $\mathbf{R}^n$. If the three points move continuously on C in such a way that they always form a proper simplex, then the circular arrow is well defined at all times, and simply slides over C following the point p. Two simplices s and t will be equivalent only if the circular arrow determined by s at s^0 can be transported over C so as to coincide with that defined by t at t^0. In particular, observe that the proper simplex $(p;q;r)$ is equivalent to the cyclically permuted copies $(q;r;p)$ and $(r;p;q)$, but not to $(r;q;p)$, $(q;p;r)$, or $(\neg p;\neg q;\neg r)$.

2.6. Orientation of a tetrahedron

A proper three-dimensional simplex $s = (o;p;q;r)$ unambiguously determines six "edges", the shortest great circle arcs connecting each pair of vertices. The orientation of the simplex can be visualized as a small corkscrew arrow, near the point o, that turns from the direction of edge oq to that of or by the shortest angle, while at the same time advancing in the direction of edge op. Alternatively, we can imagine a small hand at the point o, with the thumb pointing towards op and the other fingers curled in the same sense as the corkscrew, from oq to or. See figure 5.

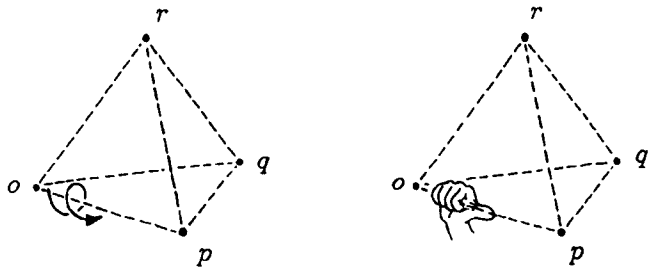

Figure 5. Orientation of simplex $(o;p;q;r)$.

The corkscrew arrow stays well-defined during any continuous deformation of the simplex, as long as the latter remains proper. If the motion is confined to a three-dimensional great sphere C, a simplex with left-threaded screw cannot be deformed into one with a right-threaded screw. In particular, the simplex $s = (o;p;q;r)$ is equivalent to $(q;r;o;p)$ and $(\neg o;\neg p;\neg q;\neg r)$, but not to $(p;q;r;o)$ or $(\neg o;p;q;r)$.

2.7. Orientation of the universe

The canonical basis of $\mathbf{R}^n$ defines the *standard simplex* $\mathbf{e} = (\mathbf{e}^0; .. \mathbf{e}^\nu)$ of $\mathbf{T}_\nu$. The point $\mathbf{e}^0 = [1, 0, \ldots, 0]$ is the *front origin*, also denoted by O, and $\mathbf{e}^1$ through $\mathbf{e}^\nu$ are the *cardinal directions*. By definition, the universe Υ_ν of $\mathbf{T}_\nu$ is oriented so that this standard simplex is positive. For example, the universe of $\mathbf{T}_2$ has the orientation of the simplex with vertices $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$. See figure 6. In the straight model, the standard simplex consists of the "vertices" of the first quadrant of the front range: the origin, the infinity point on the x-axis, and the infinity point on the y-axis, in that order.

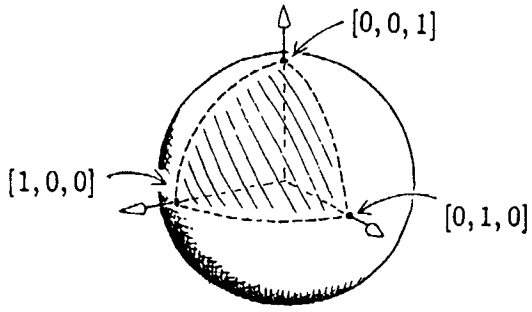

Figure 6. The standard simplex of $\mathbf{T}_2$.

3. Point location relative to a simplex

3.1. Segments

In section 2.4 we mentioned the shortest great circle arc connecting two distinct and non-antipodal points p, q of $\mathbf{T}_\nu$. It is quite natural to define *the segment* pq as being the set of points on this arc. In other words, x is on the (open) segment pq if and only if the simplices $(p; x)$ and $(x; q)$ are proper and equivalent to $(p; q)$. This set is empty if $p = q$ or $p = \neg q$. See figure 7.

3.2. The interior of a simplex

We can generalize the notion of segments to higher dimensions as follows. If $s = (s^0; .. s^\kappa)$ is a proper κ-dimensional simplex of $\mathbf{T}_\nu$, then we define the *interior of simplex* s as the set of all points x which produce a proper simplex equivalent to s when substituted for any of its vertices. That is, x is in the interior of s if and only if the simplex $(s^0; .. s^{i-1}; x; s^{i+1}; .. s^\kappa)$ is equivalent to s for all i.

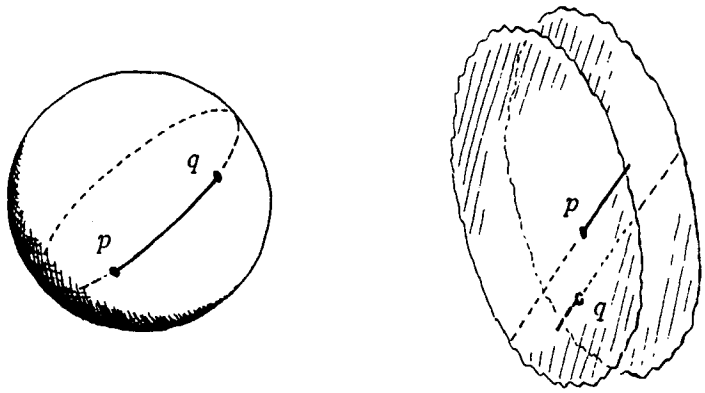

Figure 7. The segment pq.

In particular, the interior of a proper three-vertex simplex $(p; q; r)$ is the set of all points x such that the turns from xp to xq, from xp to xr, and from xr to xp are in the same direction as the turn from pq to pr. I will call this set the (*open*) *triangle* pqr. See figure 8.

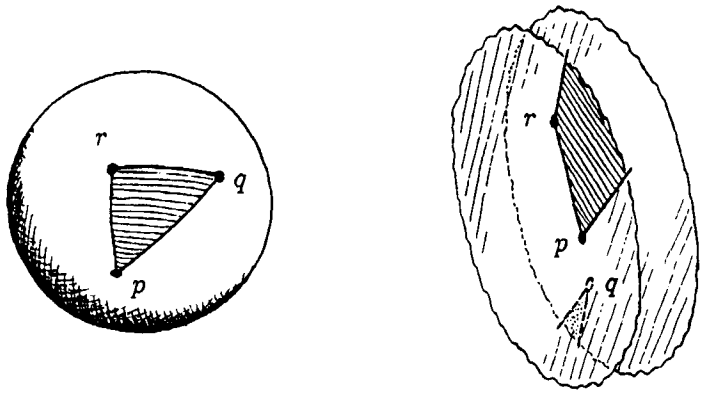

Figure 8. The triangle pqr.

3.3. Locating a point in a simplex

More generally, if x is a point on the flat spanned by a simplex $s = (s^0; .. s^\kappa)$, we can classify the position of x with respect to s by substituting x for each vertex of s in turn, and comparing the orientation of the result with that of s. The outcomes of those tests can be represented by a string $\sigma_0\sigma_1 \ldots \sigma_\nu$ of signs: $\sigma_i = +1$ (or simply '+') if replacing the ith vertex by x produces an equivalent simplex, $\sigma_i =$ '$-$' if it produces one with opposite orientation, and $\sigma_i = 0$ if it produces a degenerate simplex. This sign sequence is the *signature* of x relative to the simplex.

For any proper simplex s, all but one of the 3^k possible sign sequences can be obtained in this way. The signature $+++\cdots+$ denotes points in the interior of the simplex. In general, a non-zero signature $\sigma_0 \cdots \sigma_\nu$ denotes the interior of the simplex $(\sigma_0 \circ s^0,\ \sigma_1 \circ s^1,\ ..\ \sigma_\nu \circ s^\nu)$. Points whose signature has only $m < k$ non-zero elements lie on the subflat spanned by the corresponding m vertices of s. In that case, the signature of x relative to the subsimplex t of s with those vertices is given by the m non-zero elements of the original signature, listed in the same order. Thus, the signature $+0+00\cdots0$ means x is in the flat spanned by s^0 and s^2, and in fact in the open segment $s^0 s^2$. The only sign sequence that cannot be realized is $000\cdots0$.

For example, consider the proper simplex (p, q, r) of figure 9. The great circles determined by each pair of vertices cut the plane $\mathbf{T}_2$ into $3^3 - 1 = 26$ regions: eight open triangles, twelve open segments, and six isolated points. The signature $+++$ is the interior of the triangle pqr; $---$ denotes its antipodal image, the triangle $\neg p\, \neg q\, \neg r$. Signatures $++0$, $+-0$, $--0$, and $-+0$ stand for the open segments pq, $p\, \neg q$, $\neg p\, \neg q$, $\neg p\, q$. The signature $+00$ is produced only by the point p itself, and -00 only by its antipode $\neg p$. And so on.

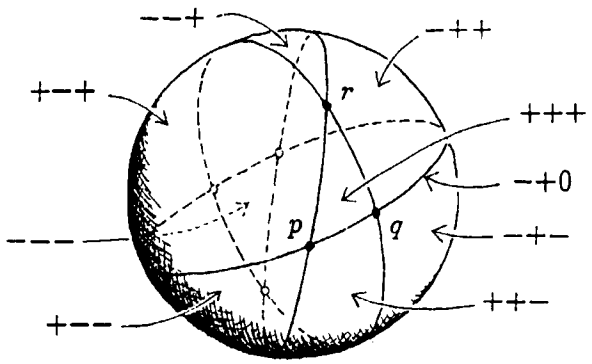

Figure 9. Signatures relative to the simplex (p, q, r).

3.4. Computing signatures

Computing the signature by the definition given above would require computing the sign of k simplices, that is, computing k determinants of order $k \times k$. In practice, a much better method is to decompose the homogeneous coordinates of x into a linear combination of the coordinates of the s^i; the signs of the coefficients of this linear combination are the desired signature. That is, if $x = [x_0, .. x_\nu]$ and

$s^i = [s^i_0, .. s^i_\nu]$, we must solve the linear system of equations

$$(\alpha_0, \cdot\cdot \alpha_\kappa) \begin{pmatrix} s^0_0 & s^0_1 & \cdots & s^0_\nu \\ \vdots & & \vdots & \\ s^\kappa_0 & s^\kappa_1 & \cdots & s^\kappa_\nu \end{pmatrix} = (x_0, x_1, \cdot\cdot x_\nu) \tag{1}$$

for the unknowns $\alpha_0, .. \alpha_\kappa$, and take $\sigma_i = \text{sign}(\alpha_i)$. The correctness of this algorithm is a trivial exercise in linear algebra. Incidentally, observe that by formula (1) the signature $00\cdots 0$ can be realized only by the null homogeneous tuple $[0, 0, .. 0]$, which is not a point of $\mathbf{T}_\nu$.

In particular, if we want to test whether a point x of $\mathbf{T}_1$ is in the segment determined by two given points p, q, we must solve

$$(\alpha_0, \alpha_1) \begin{pmatrix} p_0 & p_1 \\ q_0 & q_1 \end{pmatrix} = (x_0, x_1) \tag{2}$$

which means computing

$$\delta = \begin{vmatrix} p_0 & p_1 \\ q_0 & q_1 \end{vmatrix} = p_0 q_1 - p_1 q_0,$$

$$\beta_0 = \begin{vmatrix} x_0 & x_1 \\ q_0 & q_1 \end{vmatrix} = x_0 q_1 - x_1 q_0, \text{ and}$$

$$\beta_1 = \begin{vmatrix} p_0 & p_1 \\ x_0 & x_1 \end{vmatrix} = p_0 x_1 - p_1 x_0.$$

Then $\text{sign}(\delta)\,\text{sign}(\beta_0)$, $\text{sign}(\delta)\,\text{sign}(\beta_1)$ is the signature of x relative to $(p; q)$.

4. The homogeneous model

When looking at simplices as bases of subspaces of $\mathbf{R}^n$, the fact that their elements have unit length is an irrelevant complication. We can get rid of this assumption by working with a fourth model of the $\mathbf{T}_\nu$.

Let's define an *oriented vector space* as a vector space with one of its two classes of equivalent bases singled out as the *positive* class. In the *homogeneous model* of $\mathbf{T}_\nu$ a κ-dimensional flat is represented by a $(\kappa+1)$-dimensional oriented subspace of $\mathbf{R}^n$. In this model, a point (0-dimensional flat) is represented by a one-dimensional

oriented subspace of $\mathbf{R}^n$. The orientation of that subspace (the set of its positive bases) consists of all positive multiples of a single vector u. We can depict that subspace as a directed line through the origin of $\mathbf{R}^n$, pointing in the same direction as u. See figure 10.

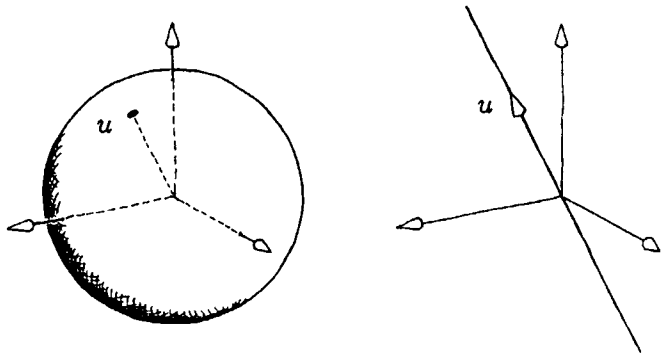

Figure 10. A point of $\mathbf{T}_2$ in the spherical and homogeneous models.

A line of $\mathbf{T}_2$ is represented by a two-dimensional linear subspace of $\mathbf{R}^3$. This can be visualized as a plane passing through the origin of $\mathbf{R}^3$, with a circular arrow on it. See figure 11.

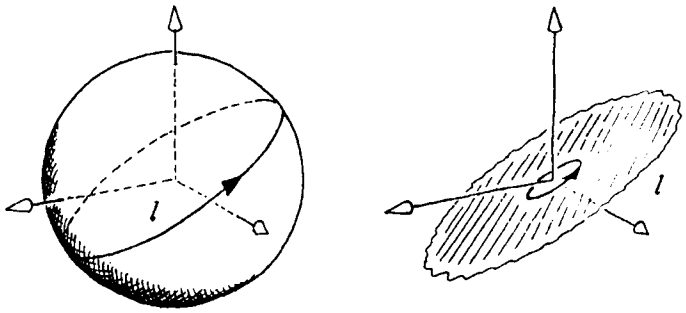

Figure 11. A line of $\mathbf{T}_2$ in the spherical and homogeneous models.

The arrow shows the direction of the shortest turn from the first to the second vector of any positive basis of that subspace. That arrow agrees with the orientation of the great circle of $\mathbf{S}_2$ representing the same line in the spherical model. In the homogeneous model, the universe Υ_ν of $\mathbf{T}_\nu$ is represented by the space $\mathbf{R}^n$ itself, with the canonical basis $\mathbf{e}^0, .. \mathbf{e}^\nu$ taken as positive.

Chapter 5
The join operation

In classical projective geometry, the join of two flats is defined as the smallest flat containing them. The join of two points is the line passing through them; the join of a point and a line is the plane containing both; and so on. This chapter defines an analogous operation in oriented projective geometry, whose arguments and result are oriented flats.

1. The join of two points

Let's consider first the join of two points in the spherical model of $\mathbf{T}_\nu$. Two points p, q generally determine a unique great circle of $\mathbf{S}_\nu$, and divide it into two unequal arcs. The *segment* pq is, by definition, the shorter of these two arcs. If we orient that great circle from p to q along the segment pq, we get the *join of* p *to* q, denoted by $p \vee q$. See figure 1.

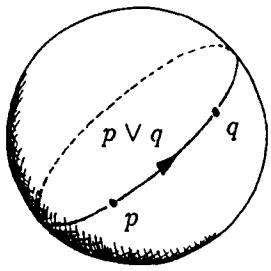

Figure 1. The join of two points.

This definition is meaningful if and only if the two points are independent, that is, $p \neq q$ and $p \neq \neg q$. Observe that $q \vee p$ is oriented in the direction opposite to that of $p \vee q$. That is, the join of two points is *anticommutative*: for all independent p, q,

$$q \vee p = \neg(p \vee q)$$

Observe also that the shortest arcs from p to q and from p to $\neg q$ leave p in opposite directions. Similarly, the arcs from p to q and from $\neg p$ to q arrive at q from opposite

directions. Therefore, for all independent pairs p, q,

$$p \vee (\neg q) = \neg(p \vee q) = (\neg p) \vee q.$$

1.1. Join in the straight model

What do the segment pq and the join $p \vee q$ look like in the straight model? If both p and q are points at infinity, the join is either $\neg\Omega$ or Ω, depending on whether the shortest turn from the direction of p to that of q (as seen from the front range) is clockwise or counterclockwise.

If at least one of the points is finite, their join is a proper line of $\mathbf{T}_\nu$, that is, two copies of the same directed Euclidean straight line (one on each range), each passing through p or $\neg p$ and q or $\neg q$. Each of the two great circle arcs connecting p and q corresponds to some subset of those two lines and their infinity points. Note that central projection does not preserve arc length, but we still can recognize the "shorter" of the two arcs as the one which does not contain any antipodal pairs.

In particular, if p and q both lie on the same range, the segment pq coincides with its Euclidean definition, and the line $p \vee q$ is oriented the obvious way. See figures 2(a) and 2(b).

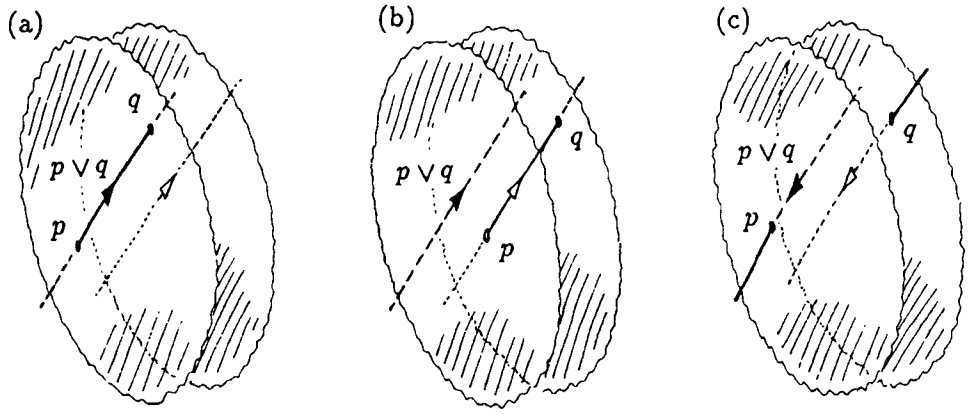

Figure 2. Join in the straight model.

If the two points are on opposite ranges, as in figure 2(c), the line $p \vee q$ is directed *away* from $\neg q$ at p, and *towards* $\neg p$ at q. Observe that the path from p to q in this direction, consisting of the ray leaving p and the one ending at q, is indeed the shortest one. The alternate path, consisting of the two complementary rays, includes all points in the Euclidean segment from p to $\neg q$ and their antipodes on the segment from $\neg p$ to q.

If p is finite but q is not, then the line $p \vee q$ is directed from p to q along the ray connecting the two. Note that if $q = d\infty$ and p is on the back range, then the direction of the line will be $-d$. See figures 3(a) and 3(b).

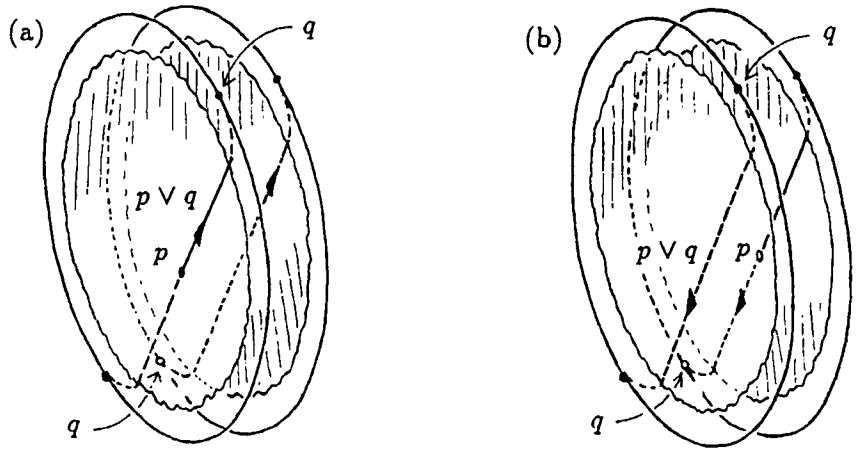

Figure 3. Join with infinity points, in the straight model.

2. The join of a point and a line

Let l be a line, and p a point not on l. The *join of p to l*, by definition, is the plane of $\mathbf{T}_n$ that contains both and is oriented so that l turns around p in the positive sense. See figure 4.

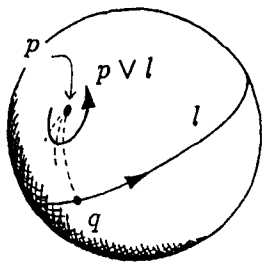

Figure 4. The join of p to l.

More precisely, let a point q move forwards on l, and observe the sense in which the line $p \vee q$ turns at l; by definition, that is the orientation of $p \vee l$.

Clearly, the orientation of a point-line join depends on the orientation of both operands:

$$p \vee (\neg l) = (\neg p) \vee l = \neg(p \vee l) \tag{1}$$

The symmetric operation $l \vee p$ can be defined as either the same as $p \vee l$ or as its opposite. It is desirable to define it in such a way as to make $\vee$ associative; that is,

$$p \vee (q \vee r) = (p \vee q) \vee r \tag{2}$$

for all $p, q, r \in \mathcal{P}$ for which those operations are defined. It turns out that both sides of equation (2) are undefined on exactly the same cases, namely when p, q, and r lie on a common line.

Equations (1) and (2) imply

$$\begin{aligned} p \vee (q \vee r) &= (p \vee q) \vee r \\ &= \neg((q \vee p) \vee r) \\ &= \neg(q \vee (p \vee r)) \\ &= q \vee (r \vee p) \\ &= (q \vee r) \vee p \end{aligned} \tag{3}$$

Since every line l can be expressed as the join of some points q and r, we conclude that

$$p \vee l = l \vee p \quad \text{for every point } p \text{ and every line } l. \tag{4}$$

Therefore, in order to make $\vee$ associative, we must make it commutative in the point-line case. At first sight this may seem a poor choice, considering that we made join anti-commutative in the point-pont case. Actually, the two definitions are quite consistent with each other. Derivation (3) above essentially says that to go from $p \vee l$ to $l \vee p$ we must reverse *two* point-point joins, and therefore the orientation of the result is not affected.

3. The join of two flats

Observe that the join of points p and q can be defined as the line containing both, oriented so that the pair p, q is a positive simplex. Moreover, if p is a point, l is a line, and $(q; r)$ is a positive simplex of l, then the join of p to l is the plane containing both, oriented so that $(p; q; r)$ is a positive simplex. I will define the join of general flats by generalizing this observation. That is,

Definition 1. The join of two flats determined by simplices u, v is the flat defined by their concatenation. That is,

$$[u^0; .. u^\kappa] \vee [s^0; .. s^\mu] = [u^0; .. u^\kappa; s^0; .. s^\mu].$$

With a little linear algebra we can easily check that the concatenation of two proper simplices is a proper simplex if and only if the corresponding flats have no point in

common. If they do, their join is undefined. It is easy to see also that the result of the join is the same no matter which simplices we choose to represent the two flats.

The join of flats with rank zero must be defined separately. By definition,

$$\begin{aligned} \Lambda \vee a &= a = a \vee \Lambda \\ (\neg\Lambda) \vee a &= \neg a = a \vee (\neg\Lambda) \end{aligned}$$

for all a. In other words, Λ is the (left and right) identity of join. Note that the flats Λ and $\neg\Lambda$ are disjoint from every flat, even from themselves; they behave like oriented empty sets (hence their names).

4. Properties of join

Note that every flat of rank $k \geq 1$ is the join of the k vertices of any of its positive simplices. Obviously, whenever $a \vee b$ is defined we have

$$\operatorname{rank}(a \vee b) = \operatorname{rank}(a) + \operatorname{rank}(b). \tag{5}$$

4.1. Associativity

The associativity of $\vee$ follows directly from the definition: we have

$$a \vee (b \vee c) = (a \vee b) \vee c \tag{6}$$

for any flats a, b, c, when either side is defined. Also,

$$(\neg a) \vee b = a \vee (\neg b) = \neg(a \vee b) \tag{7}$$

for all disjoint flats a, b.

4.2. Commutativity

Recall that the join of two points depends on the order of the operands, whereas that of a point and a line does not. The general rule follows readily from definition 1. Observe that transposing the order of two vectors in a basis reverses its orientation. Observe also that in going from $[a^0; .. \, a^\kappa; \, b^0; .. \, b^\mu]$ to $[b^0; .. \, b^\mu; \, a^0; .. \, a^\kappa]$ we have to transpose $(\kappa + 1)(\mu + 1)$ adjacent pairs of vectors. We conclude that for any two flats a, b,

$$a \vee b = \neg^{\operatorname{rank}(a)\operatorname{rank}(b)}(b \vee a) \tag{8}$$

That is, reversing the order of the operands in a join reverses the orientation of the result as many times as the product of the ranks of those operands. Therefore, $a \vee b = b \vee a$ if one of the operands has even rank, and $a \vee b = \neg(b \vee a)$ if both have

odd rank. For example, the join of a point and a line is commutative, but that of a point and a plane is not. Observe that equations (6), (7), and (8) are valid even for flats of rank zero.

5. Null objects

Recall that $a \vee b$ is undefined if a and b have a common point. It may be tempting to plug this hole, and extend the definition so that $a \vee b$ is always one of the two smallest flats containing a and b. This extension is common in classical projective geometry, but unfortunately it cannot be made to work in the oriented framework. If a and b are not disjoint, it is impossible to define the orientation of $a \vee b$ in a consistent way.

We can understand the difficulty as follows. If we apply the commutativity law (8) to the expression $p \vee p$, we get

$$p \vee p \;=\; \neg(p \vee p)$$

which cannot be true for any point, or indeed for any flat. This shows we cannot consistently define $p \vee p$. In general, if two flats a and b have a common point p, we can always write them as $a = u \vee p$ and $b = p \vee v$ for some (possibly vacuous) flats u, v. Then by associativity we must have

$$a \vee b \;=\; u \vee (p \vee p) \vee v$$

We conclude that $a \vee b$ cannot be consistently defined when a and b are not disjoint.

This problem is not as serious as it may seem. Even in unoriented geometry the extended join operation cannot be made continuous, since the dimension of the result may change abruptly with infinitesimal changes in the operands. In most geometry algorithms, those degenerate cases require special handling anyway, so the proposed extension would not make programs much simpler. The extended join may actually be a nuisance in strongly typed languages such as Pascal and Modula-2, where one usually wants to declare objects of different dimensions as having different compile-time types.

Nevertheless, from the programmer's viewpoint partially defined operations are bothersome. It is generally preferable to make them total, by adding a new "undefined" element to their range, and letting this be the result of the operation whenever it was not defined originally. Accordingly, I will postulate a dummy *null object* $\mathbf{0}^k$ for every integer k, and let $a \vee b$ be $\mathbf{0}^{\mathrm{rank}(a)+\mathrm{rank}(b)}$ whenever a and b are not disjoint.

As we shall see, this extension is quite natural from the computational point of view, and can be implemented at zero or negative cost: the same code that

produces $a \vee b$ in the normal case will produce the undefined flat $\mathbf{0}^k$ if a and b are not disjoint. In fact, a practical test for whether a and b intersect is to compute $a \vee b$ by the standard algorithm, and check whether the result is $\mathbf{0}^k$.

It is convenient to let also $\neg\mathbf{0}^k = \mathbf{0}^k$, and $\mathbf{0}^k \vee a = a \vee \mathbf{0}^k = \mathbf{0}^k \vee \mathbf{0}^m = \mathbf{0}^{k+m}$ for all flats a of rank m. With these rules, all properties of join listed so far are always true, even when the operands are not disjoint and/or are null objects. However, note that $\mathbf{0}^k$ is quite unlike ordinary flats in many respects; for example, it fails to satisfy $a \neq \neg a$. For that reason, I will neither call it a flat nor include it in $\mathcal{F}$.

Usually, the rank of null objects is irrelevant or known from the context, so I will write simply $\mathbf{0}$ instead of $\mathbf{0}^k$. However, in computer implementations (especially in strongly-typed languages) it is usually simpler to to use different representations for the null object of each rank.

6. Complementary flats

An important result, that is easily proved by reference to the homogeneous model, is

Theorem 1. *For any subflat x of a flat a there is a flat y such that $x \vee y = a$.*

PROOF: Let X and A be the oriented vector spaces representing x and a in the homogeneous model. Since x is a subflat of a, the space X is a subspace of A. From linear algebra we know that there is a basis $s = (s^0; .. s^\kappa)$ for A whose first m elements ($m =$ rank of x) are a basis for X.

Let u be the flat determined by the first m elements of s, and v the one determined by the last $k - m$. Because of the way s was selected, u is either x or $\neg x$, and $u \vee v$ is either a or $\neg a$. It follows that either $x \vee v = a$ or $x \vee (\neg v) = a$. QED.

A flat y with this property is said to be a *right complement of x in a*. (Symmetrically, the flat x is said to be a *left complement* of x in a.) Note that the (right and left) complement of a itself is Λ, and that of $\neg a$ is $\neg\Lambda$; and vice-versa. Except for these special cases, the right and left complements are not unique.

Chapter 6
The meet operation

The intersection of two lines to give a point is the second fundamental operation of classical geometry. We can also intersect a line and a plane to get a point, two planes to get a line, and in general two flats to get a flat.

Unoriented geometry considers flats to be sets of points, and therefore their intersection doesn't have to be specially defined: it is plain set intersection. In oriented geometry, we must also choose an orientation for the resulting set; a consistent way to do this is given by the *meet* operation defined below. The meet operation has properties similar to those of join, and is in fact its dual, in a very precise sense.

1. The meeting point of two lines

For example, two lines of $\mathbf{T}_2$ generally intersect on a pair of antipodal points. See figure 1. To choose an orientation for the intersection means to pick one member of the pair as *the* meeting point of the two lines.

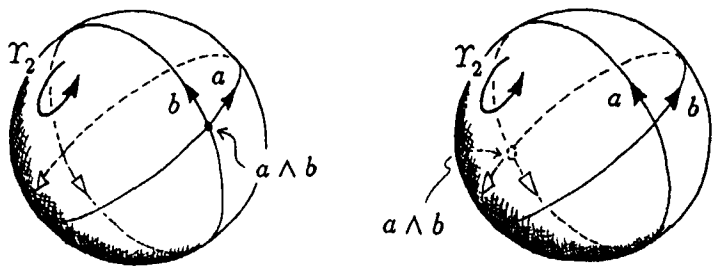

Figure 1. The meet of two lines.

Note that the shortest turn from the direction of a to that of b is positive at one of the two common points, and negative at the other. By definition, the first one is the *point where a meets b*, denoted by $a \wedge b$. More precisely, if u_a and u_b are vectors tangent to a and b (and agreeing with their directions) at $p = a \wedge b$, then p, u_a, and u_b (in that order) are a positive basis of $\mathbf{R}^3$. Note that at the antipodal point all three vectors are exactly reversed, and therefore form a basis of opposite handedness. The meet $a \wedge b$ is not defined when $a = b$ or $a = \neg b$.

Like the join of two points, the meet of two lines is anticommutative, and depends on the orientation of its operands. For any two lines a, b of $\mathbf{T}_2$, we have

$$b \wedge a = \neg(a \wedge b)$$

$$(\neg a) \wedge b = a \wedge (\neg b) = \neg(a \wedge b)$$

This behavior is unavoidable if the meet operation is to be continuous and defined for any two unrelated lines. To see why, consider two lines a and b (on the spherical model of $\mathbf{T}_2$) that intersect at a point p in such a way that b is 90° counterclockwise from a. See figure 2(a).

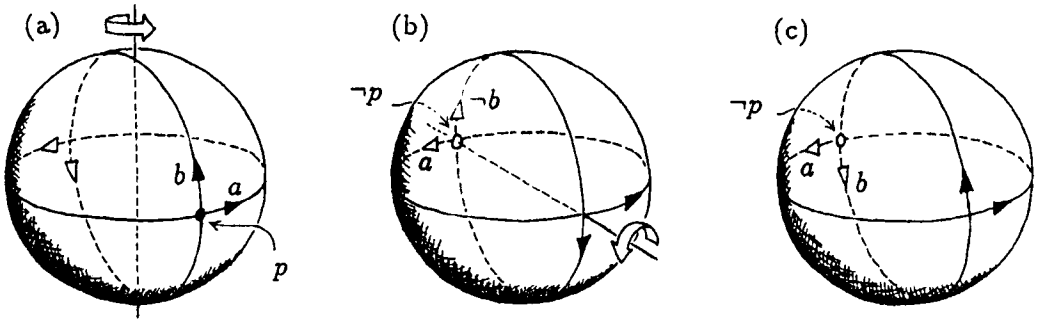

Figure 2.

Imagine that the sphere rotates by 180° around an axis perpendicular to the plane of a. This continuous motion keeps a fixed, but takes b to $\neg b$ and exchanges p with $\neg p$. See figure 2(b). Therefore, no matter which of the two points we define to be $a \wedge b$, we must have $a \wedge (\neg b) = \neg(a \wedge b)$.

Now start from the situation in figure 2(b), and rotate the sphere by 90° around the line of the two intersection points, so as to take a into b and $\neg b$ into a. See figure 2(c). By continuity, we conclude that $b \wedge a = a \wedge (\neg b)$, which we have just shown to be $\neg(a \wedge b)$.

1.1. The relativity of meet

Note that the meet of two lines as defined above depends strongly on the orientation of the whole plane $\mathbf{T}_2$. This is not merely a weakness of our definition, but rather an essential property of oriented intersections. It turns out that it is not possible to consistently select one of the intersection points without a reference orientation for the whole plane.

To see why, consider two intersecting finite lines a, b of $\mathbf{T}_3$ (in the straight model). Let l be the bisecting line of the angle ab. Now rotate a and b by 180° around the axis l. This continuous motion exchanges a with b, while keeping the intersection points fixed and avoiding degeneracies ($a = b$ or $a = \neg b$). By continuity, we should then have $b \wedge a = a \wedge b$. But this contradicts our previous proof that $b \wedge a = \neg(a \wedge b)$. Therefore, if a and b are intersecting lines in three-space, any definition of $a \wedge b$ must be ambiguous, or must be discontinuous for some pairs a, b with $a \neq b$ and $a \neq \neg b$.

Therefore, we cannot define the oriented meet of two coplanar lines in $\mathbf{T}_3$ or in a higher-dimensional space, since (as we saw in the previous chapter) there is no consistent way to pick an orientation for the plane containing them. In general, the meet of two flats cannot be defined in an absolute sense, but only relative to some oriented flat of suitable dimension that contains them.

2. The general meet operation

The meeting point of two lines in $\mathbf{T}_2$ can also be defined by the following expression: for any three points p, q, r of $\mathbf{T}_2$,

$$p \vee q \vee r = \mathbf{T}_2 \;\Leftrightarrow\; (p \vee q) \wedge (q \vee r) = q \tag{1}$$

See figure 3.

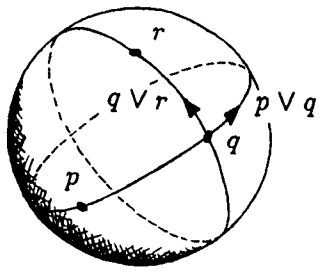

Figure 3. The meet of two lines in $\mathbf{T}_2$.

We can define the meet of two arbitrary flats by a straightforward extension of formula (1):

Definition 1. If u is a flat of minimum rank enclosing flats a and b, then the *meet of a and b in u* is the flat y such that

$$\begin{aligned} &x \vee y \vee z = u, \\ &x \vee y = a, \text{ and} \\ &y \vee z = b, \end{aligned} \tag{2}$$

for some $x, z \subseteq u$. I will denote the meet y by $a \wedge_u b$.

In other words, for any three flats x, y, z such that $x \vee y \vee z = u \neq \mathbf{0}$, we have by definition

$$(x \vee y) \wedge_u (y \vee z) = y \tag{3}$$

Before we go on, we must prove that definition 1 is consistent: that is, we must show that such a flat y always exists and is unique. The next lemma shows this is indeed the case:

Theorem 1. *If u is a flat of minimum rank containing flats a and b, then there are some flats x, z, and a unique flat y satisfying equations* (2).

PROOF: The intersection of a and b, viewed as sets of points, is some unoriented flat contained in u. Let y be any oriented version of that flat. Since $y \subset a$, it has a left complement in a: that is, there is flat x such that $x \vee y = a$. Similarly, there is a flat z such that $y \vee z = b$.

Since z is contained in b and disjoint from $y = a \cap b$, we conclude z is disjoint from a, and therefore the flat $v = x \vee y \vee z = a \vee z$ is well-defined. Now x, y, and z are contained in u, so v too is contained in u. Also, v contains $x \vee y = a$ and $y \vee z = b$. Since u is a flat of minimum rank containing a and b, v must be either u or $\neg u$. Then either the flats x, y, z or the flats $\neg x, \neg y, \neg z$, respectively, will satisfy (2).

To show the uniqueness of y, let x,y, and z be defined as above, with orientations reversed as needed to make equations (2) hold. Suppose that equations (2) are satisfied also by flats x', y', z'. Since $a = x' \vee y'$ and $b = y' \vee z'$, we must have $y' \subseteq a \cap b$, and $x' \subseteq a \setminus b$. Since $a \cap b$ is the set of points y, we conclude

$$y' \subseteq y. \tag{4}$$

We also conclude $x' \cap y = \emptyset$, which means $x' \vee y$ is defined; since this flat is contained in a, we must have $\text{rank}(x') + \text{rank}(y) \leq \text{rank}(a)$. From this and from $\text{rank}(x') + \text{rank}(y') = \text{rank}(x' \vee y') = \text{rank}(a)$ we conclude $\text{rank}(y') \geq \text{rank}(y)$. Together with (4) this implies $y' = y$ or $y' = \neg y$. Now suppose $y' = \neg y$. Then

from $x \vee y = a = x' \vee y'$ and $y \vee z = b = y' \vee z'$ we get

$$\begin{aligned} x \vee y \vee z &= x' \vee y' \vee z \\ &= x' \vee (\neg y) \vee z \\ &= \neg(x' \vee y \vee z) \\ &= \neg(x' \vee y' \vee z'), \end{aligned}$$

contradicting the assumption that $x \vee y \vee z = x' \vee y' \vee z'$.
QED.

Incidentally, this proof shows also that the meet of two flats is simply the intersection of their point sets, oriented in a specific way.

I will omit the subscript in $\wedge_u$ when the reference flat u (the *universe* of the operation) is implied by the context. In particular, I will use the phrases *the meet operation of* $\mathbf{T}_\nu$ or *the ν-dimensional meet* to signify the meet operation relative to the standard universe Υ_ν of $\mathbf{T}_\nu$.

2.1. Null objects

Note that definition 1 specifies $a \wedge_u b$ only if u is the smallest flat enclosing both a and b; which is to say, if and only if $\text{rank}(a) + \text{rank}(b) - \text{rank}(a \cap b) = \text{rank}(u)$. When that is not the case, $a \wedge_u b$ is undefined. As in the case of join, it is convenient to extend meet to a total function anyway, by letting $a \wedge b$ to be the null object $\mathbf{0}^{\text{rank}(a)+\text{rank}(b)-\text{rank}(u)}$, whenever the result is not specified by definition 1. I will also postulate $\mathbf{0}^k \wedge a = a \wedge \mathbf{0}^k = \mathbf{0}^{k+\text{rank}(a)-\text{rank}(u)}$, for all a. As in the case of join, this convention can be implemented at zero or negative cost.

3. Meet in three dimensions

3.1. The meet of a line and a plane

To illustrate the definition, let's consider some examples in $\mathbf{T}_3$. For example, a line l and a plane π generally have two antipodal points in common. According to the definition, $\pi \wedge l = x$ if and only if there are points p, q on π and r on l such that $(p; q; x)$ is a positive triangle of π, $(x; r)$ is a positive pair on l, and $(p; q; x; r)$ is a positive tetrahedron of $\mathbf{T}_3$. We conclude that $\pi \wedge l$ is the point where the circular arrow of π and the longitudinal arrow of l are like the fingers and thumb of the right

hand. See figure 4.

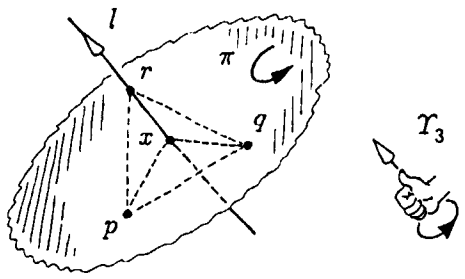

Figure 4. The meet of a line and a plane.

By an entirely analogous argument, we conclude that $l \wedge \pi$ is the same point as $\pi \wedge l$; that is, the meet of a line and a point is commutative. The result of $\pi \wedge l$ is undefined if l is contained in π.

3.2. The meet of two planes

The intersection of two planes π, σ in $\mathbf{T}_3$ is a pair of oppositely oriented lines. According to definition 1, we must find points p, q, r, s such that $(p; q; r)$ is a positive triangle of π, $(q; r; s)$ is a positive triangle of σ, and $(p; q; r; s)$ is a positive tetrahedron of $\mathbf{T}_3$. Then $\pi \wedge \sigma$ will be the line from q to r. See figure 5.

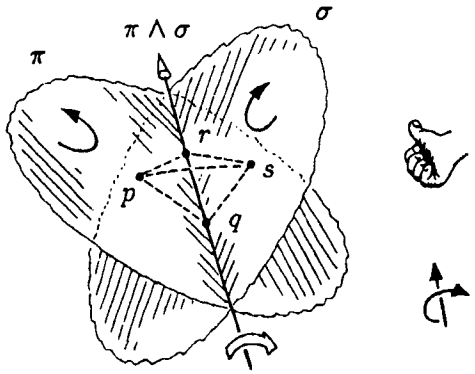

Figure 5. The meet of two planes in $\mathbf{T}_3$.

Informally, we must imagine π turning towards σ around their common line, by the smallest angle that makes the two planes coincide in position and orientation. Then we can figure out the direction of $\pi \wedge \sigma$ from this turning direction by the right-hand rule. Alternatively, if we look at the intersection of the two planes from the side where π is at our left and oriented counterclockwise, and σ is to our right oriented

clockwise, then $\pi \wedge \sigma$ will be directed upwards. Note that $\sigma \wedge \pi = \neg(\pi \wedge \sigma)$; that is, the meet of two planes in $\mathbf{T}_3$ is anticommutative.

4. Properties of meet

Note that the orientation of the result depends on that of the reference flat u, as well as on those of a and b. In fact, if we replace x, y, and z by their opposites in the equations (2), we conclude that

$$a \wedge_{(\neg u)} b = \neg(a \wedge_u b) \tag{5}$$

In what follows $\wedge$ denotes the meet operation relative to some fixed ν-dimensional flat Υ (which may or may not be the standard universe of $\mathbf{T}_\nu$).

4.1. Orientation reversal

It follows from the definition that for any flats or null objects a, b,

$$(\neg a) \wedge b \;=\; a \wedge (\neg b) \;=\; \neg(a \wedge b). \tag{6}$$

4.2. Meet with universe

The reference flat Υ acts as the unit element of $\wedge$: for all flats $a \subseteq \Upsilon$,

$$\begin{aligned} a \wedge \Upsilon \;&=\; a \;=\; \Upsilon \wedge a, \\ a \wedge (\neg\Upsilon) \;&=\; \neg a \;=\; (\neg\Upsilon) \wedge a. \end{aligned} \tag{7}$$

4.3. Meet in different spaces

The relationship between the meet operation of different universes is illuminated by the next lemma, which follows immediately from the definition:

Lemma 2. *Let x, y, and u be flats such that $x \vee u \vee y \neq \mathbf{0}$. Then for any $a, b \subseteq u$ we have*

$$(x \vee a) \;\wedge_{x \vee u \vee y}\; (b \vee y) \;=\; a \wedge_u b \tag{8}$$

In particular, by taking $x = \Lambda$ or $y = \Lambda$ we conclude that, if $x \vee u \neq \mathbf{0}$, then

$$(x \vee a) \;\wedge_{x \vee u}\; b \;=\; a \wedge_u b \tag{9}$$

$$a \;\wedge_{u \vee x}\; (b \vee x) \;=\; a \wedge_u b \tag{10}$$

for all $a, b \subseteq u$. Among other things, this allows us to establish a connection between the meet operations in $\mathbf{T}_2$ and $\mathbf{T}_3$. Let's consider $\mathbf{T}_2$ embedded as a plane of $\mathbf{T}_3$ in the standard way: that is, let's identify the point $[w, x, y]$ of $\mathbf{T}_2$ with $[w, x, y, 0]$ of $\mathbf{T}_3$. With these conventions we can write $\mathbf{T}_3 = \mathbf{T}_2 \vee \mathbf{e}^3$, where $\mathbf{e}^3 = [0, 0, 0, 1]$ is the point at plus infinity on the front z-axis of $\mathbf{T}_3$. Then, by lemma 2, the meet of lines a and b (relative to $\mathbf{T}_2$) is the meet of the line a and the vertical plane $b \vee \mathbf{e}^3$ (relative to $\mathbf{T}_3$). See figure 6.

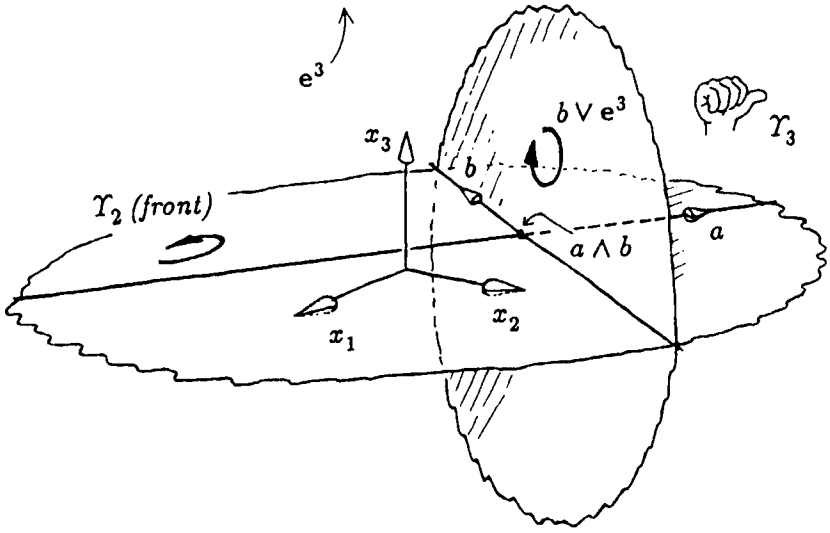

Figure 6. Relation between meet operation in $\mathbf{T}_2$ and $\mathbf{T}_3$.

4.4. Co-ranks

When working with the meet operation relative to a fixed universe Υ, it is convenient to classify the flats of Υ by their *complementary rank* or *co-rank*, defined by $\text{corank}(a) = \text{rank}(\Upsilon) - \text{rank}(a) = \dim(\Upsilon) - \dim(a)$. In particular, the reference flat Υ itself has co-rank 0, and its hyperplanes have co-rank 1. The vacuum has co-rank equal to the rank of Υ. In general, the co-rank of a flat a is how many points must be joined to a to get the universe. From the definition of meet it follows that

$$\text{corank}(a \wedge b) = \text{corank}(a) + \text{corank}(b)$$

and

$$\text{rank}(a \wedge b) = \text{rank}(a) - \text{corank}(b) = \text{rank}(b) - \text{corank}(a)$$

That is, the meet operation lowers the rank of one operand by the co-rank of the other. Since hyperplanes of Υ have co-rank equal to one, we conclude that the co-

rank of a flat a is also the number of hyperplanes we have to meet in order to get a.

4.5. Commutativity

The general meet operation is either commutative or anti-commutative, depending on the ranks of the operands and the rank of the reference space. More precisely,

Theorem 3. *For all flats a and b,*

$$b \wedge a = \neg^{\operatorname{corank}(a)\operatorname{corank}(b)}(a \wedge b) \tag{11}$$

whenever $a \wedge b$ is defined.

PROOF: If either side is the null object, then the theorem is trivial. Otherwise, there must be flats x, y, z such that $x \vee y \vee z = \Upsilon$, $x \vee y = a$, and $y \vee z = b$, with $y = a \wedge b$. Let r, s, and t be the ranks of x, y and z, respectively. Then $z \vee y \vee x = \neg^{rs+rt+st}\Upsilon$. It follows that $(z \vee y) \wedge (y \vee x) = \neg^{rs+rt+st}y$. But $z \vee y = \neg^{st}b$, and $y \vee x = \neg^{rs}a$. Therefore, $b \wedge a = \neg^{rt}(a \wedge b)$. Since $r + s + t = \operatorname{rank}(\Upsilon)$ and $r + s = \operatorname{rank}(a)$, it follows that $r = \operatorname{corank}(b)$. Similarly $t = \operatorname{corank}(a)$, and this concludes the proof.

QED.

Theorem 3 says that $\wedge$ is commutative, unless both operands have odd co-rank. In three dimensions and less, the only "odd" cases are: two points in $\mathbf{T}_1$, two lines in $\mathbf{T}_2$, two planes or a point and a plane in $\mathbf{T}_3$.

4.6. Associativity

Like join, meet is associative. In order to prove this fact, we will need the following useful lemma, which provides an alternative to definition 1:

Lemma 4. *Let u be any flat. Then, for all $x, y, z \subseteq u$,*

$$\begin{aligned} x \vee y \vee z = u &\Leftrightarrow (x \vee y) \wedge_u (x \vee z) = x \\ &\Leftrightarrow (x \vee z) \wedge_u (y \vee z) = z \end{aligned} \tag{12}$$

PROOF: Let $r = \operatorname{rank}(x)\operatorname{rank}(y)$. By the commutativity properties of join and by definition 1, we have

$$\begin{aligned} x \vee y \vee z = u &\Leftrightarrow y \vee x \vee z = \neg^r u \\ &\Leftrightarrow (y \vee x) \wedge (x \vee z) = \neg^r x \\ &\Leftrightarrow (x \vee y) \wedge (x \vee z) = x. \end{aligned}$$

The proof for $(x \vee z) \wedge_u (y \vee z) = z$ is entirely symmetric.
QED.

It is important to notice that not every permutation of x, y, and z in the right-hand side of equation (12) will make the formulas true. For example, if $x \vee y \vee z = u$, it doesn't follow that $(y \vee z) \wedge_u (x \vee z) = z$.

Lemma 5. *For any flats* a, b, c,

$$(a \wedge b) \wedge c = \mathbf{0} \quad \Leftrightarrow \quad a \wedge (b \wedge c) = \mathbf{0} \tag{13}$$

PROOF: If either of a, b, or c is a null object, the theorem is obvious. Otherwise, if $a \wedge b = \mathbf{0}$, a and b must be contained in some flat f with rank less than n; then a and $b \wedge c$ are also contained in f, and therefore both sides of equation (13) are null. The case $b \wedge c = \mathbf{0}$ is entirely symmetrical. So, assume $a \wedge b$ and $b \wedge c$ are both defined; we must have

$$\text{rank}(a) + \text{rank}(b) - \text{rank}(a \wedge b) = n \tag{14}$$
$$\text{rank}(b) + \text{rank}(c) - \text{rank}(b \wedge c) = n \tag{15}$$

Subtracting (15) from (14) we get

$$\text{rank}(a) + \text{rank}(b \wedge c) = \text{rank}(a \wedge b) + \text{rank}(c)$$

Hence,

$$\text{rank}(a) + \text{rank}(b \wedge c) - \text{rank}(a \cap (b \wedge c)) = n$$

if and only if

$$\text{rank}(a \wedge b) + \text{rank}(c) - \text{rank}((a \wedge b) \cap c) = n,$$

which means $a \wedge (b \wedge c)$ is defined if and only if $(a \wedge b) \wedge c$ is defined.
QED.

Lemma 6. *For any flats* a, b, c,

$$(a \wedge b) \wedge c = \Lambda \quad \Leftrightarrow \quad a \wedge (b \wedge c) = \Lambda.$$

PROOF: Suppose $(a \wedge b) \wedge c = \Lambda$. By lemma 5, $a \wedge (b \wedge c)$ must be defined. Since meet is an oriented intersection, and intersection is associative, $a \wedge (b \wedge c)$ must be either Λ or $\neg\Lambda$. Also, $a \wedge c$ must be defined; for, if there were some flat of less than full rank containig a and c, it would also contain $a \wedge b$ and c, and $(a \wedge b) \wedge c$

would be **0**. So, let

$$x = a \wedge b$$
$$y = a \wedge c$$
$$z = b \wedge c$$

I claim $x \vee y = a$, $x \vee z = b$, $y \vee z = c$, and $x \vee y \vee z = \Upsilon$. First, since $x \cap y = a \cap b \cap c = \emptyset$, the flat $x \vee y$ is defined. Since x and y are flats of a, $x \vee y \subseteq a$ Also, since $\text{rank}((a \wedge b) \wedge c) = 0$, we deduce $\text{corank}(c) = \text{rank}(a \wedge b)$, and

$$\begin{aligned}\text{rank}(x \vee y) &= \text{rank}((a \wedge b) \vee (a \wedge c)) \\ &= \text{rank}(a \wedge b) + \text{rank}(a \wedge c) \\ &= \text{corank}(c) + (\text{rank}(a) - \text{corank}(c)) \\ &= \text{rank}(a)\end{aligned}$$

which means $x \vee y = \alpha \circ a$ for some $\alpha \in \{\pm 1\}$. In the same way, we can show that $x \vee z = \beta \circ b$ and $y \vee z = \gamma \circ c$ for some $\gamma \in \{\pm 1\}$.

The hypothesis $(a \wedge b) \wedge c = \Lambda$ implies $(a \wedge b) \vee c = \Upsilon$, and, therefore,

$$x \vee y \vee z = (a \wedge b) \vee (\gamma \circ c) = \gamma \circ (a \wedge b) \vee c = \gamma \circ \Upsilon. \tag{16}$$

By the definition of meet and by lemma 4 it follows from (16) that

$$\begin{aligned}(x \vee y) \wedge (y \vee z) &= \gamma \circ y = \gamma \circ (a \wedge c), \\ (x \vee y) \wedge (x \vee z) &= \gamma \circ x = \gamma \circ (a \wedge b), \\ (x \vee z) \wedge (y \vee z) &= \gamma \circ z = \gamma \circ (b \wedge c).\end{aligned} \tag{17}$$

On the other hand,

$$\begin{aligned}(x \vee y) \wedge (y \vee z) &= (\alpha \circ a) \wedge (\gamma \circ c) = \alpha\gamma \circ (a \wedge c), \\ (x \vee y) \wedge (x \vee z) &= (\alpha \circ a) \wedge (\beta \circ b) = \alpha\beta \circ (a \wedge b), \\ (x \vee z) \wedge (y \vee z) &= (\beta \circ b) \wedge (\gamma \circ c) = \beta\gamma \circ (b \wedge c).\end{aligned} \tag{18}$$

Comparing (17) and (18) we conclude $\alpha = \beta = \gamma = +1$. Therefore we have $x \vee y = a$ and $x \vee y \vee z = \Upsilon$, which means

$$a \wedge (b \wedge c) = (x \vee y) \wedge z = \Lambda.$$

The converse follows from this and the commutativity law.
QED.

We are now ready to prove the main result:

Theorem 7. *Meet is associative: for any three flats a, b, c,*

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \tag{19}$$

whenever one of the two expressions is defined.

PROOF: If one side of (19) is $\mathbf{0}$, then by lemma 5 the other side is $\mathbf{0}$, too. So, assume both sides are defined. Let $w = (a \wedge b) \wedge c$. Since w is a subflat of a, b, and c, there are flats t, u, v such that

$$\begin{aligned} a &= w \vee t, \\ b &= w \vee u, \\ c &= w \vee v. \end{aligned}$$

Also, there is a flat s such that $w \vee s = \Upsilon$. Therefore, by equation (9) we have

$$\begin{aligned} (a \wedge b) \wedge c &= \big((w \vee t) \wedge_{w \vee s} (w \vee u)\big) \wedge_{w \vee s} (w \vee v) \\ &= \big(w \vee (t \wedge_s u)\big) \wedge_{w \vee s} (w \vee v) \\ &= w \vee ((t \wedge_s u) \wedge_s v) \end{aligned}$$

Similarly,

$$a \wedge (b \wedge c) = w \vee (t \wedge_s (u \wedge_s v)) \tag{20}$$

From $w = (a \wedge b) \wedge c$ and equation (20) we conclude that $(t \wedge_s u) \wedge_s v = \Lambda$. By lemma 6, this implies $t \wedge_s (u \wedge_s v) = \Lambda$. Then equation (20) says

$$a \wedge (b \wedge c) = w \vee \Lambda = w = (a \wedge b) \wedge c.$$

QED.

Chapter 7
Relative orientation

One advantage of two-sided geometry is that it allows us to talk about the two sides of a line in the plane, or of a plane in three-space. As shown below, these concepts can be elegantly expressed in terms of the join and meet operations.

1. The two sides of a line

A line l divides the spherical model of the two-sided plane in two halves. I call these the *left* and *right* (or *positive* and *negative*) sides of l, as they would be seen by an ant crawling along the line on the outside of the sphere. See figure 1.

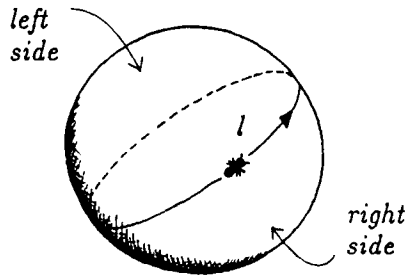

Figure 1. The two sides of a line of $\mathbf{T}_2$.

More precisely, let $(q; r)$ be any positive simplex of the line. I will say that a point p is on the *positive* (or *left*) *side* of l if the simplex $(p; q; r)$ (in that order) is a positive triangle of $\mathbf{T}_2$. Symmetrically, p is on the *negative* (*right*) side of l if $(p; q; r)$ is a negative triangle.

1.1. The sides of a line in the straight model

How are these concepts mapped to the straight model? If the line m is at infinity, its left and right sides are the front and back ranges of $\mathbf{T}_2$ (when $m = \Omega$), or vice-versa (when $m = \neg\Omega$). If m is a finite line, the picture is a bit more involved. Let d be the direction of m, L^+ and R^+ be the left and right half-planes determined by m on the front range, and L^-, R^- be their antipodal images on the back range. Then the left side of m is the union of L^+, R^-, and the infinities $u\infty$ for u in the

counterclockwise arc from $+d$ to $-d$. The right side of m consists of R^+, L^-, and the supplementary arc on the line at infinity. See figure 2. The reversal of left and right on the back range is to be expected, since the meaning of "counterclockwise" is reversed there, but the longitudinal orientation of lines is not.

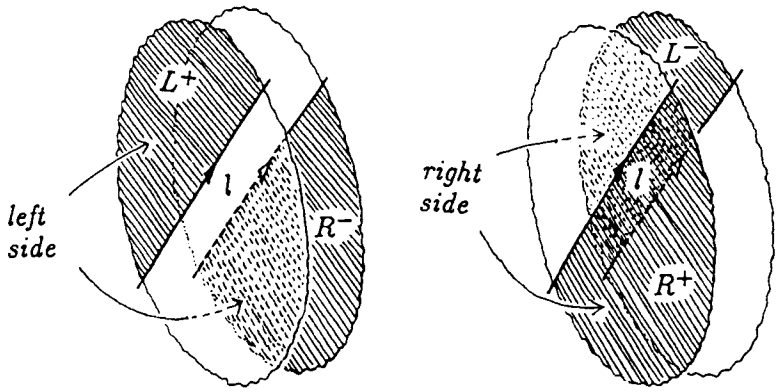

Figure 2. The two sides of a line, in the straight model of $\mathbf{T}_2$.

1.2. Connection to join and meet

We can define the left and right sides of a line in terms of join and meet, as follows:

$$p \text{ is } \left\{ \begin{array}{c} \text{to the left of } l \\ \text{on } l \\ \text{to the right of } l \end{array} \right\} \text{ iff } p \vee l = \left\{ \begin{array}{c} \Upsilon_2 \\ \mathbf{0}^2 \\ \neg\Upsilon_2 \end{array} \right\} \text{ iff } p \wedge l = \left\{ \begin{array}{c} \Lambda \\ \mathbf{0}^0 \\ \neg\Lambda \end{array} \right\} \qquad (1)$$

Notice how this definition derives the "transversal" (left-right) orientation of l by combining the "longitudinal" orientation of l with the intrinsic "circular" orientation of the universe Υ_2.

2. Relative position of arbitrary flats

The same idea can be used to distinguish the two half-spaces determined by a hyperplane of $\mathbf{T}_\nu$, for arbitrary ν. In fact, we can generalize (1) to a test of the relative placement of any two flats a and b of $\mathbf{T}_\nu$ whose ranks add to n. I define the *relative orientation of a and b* as the sign-valued function

$$a \diamond b = \left\{ \begin{array}{rl} +1 & \text{if } a \vee b = \Upsilon_\nu, \\ 0 & \text{if } a \vee b = \mathbf{0}^n. \\ -1 & \text{if } a \vee b = \neg\Upsilon_\nu, \end{array} \right.$$

From the definitions of $\wedge$ and $\diamond$, it is obvious that

$$a \diamond b = \begin{cases} +1 & \text{if } a \wedge b = \Lambda, \\ 0 & \text{if } a \wedge b = \mathbf{0}^0. \\ -1 & \text{if } a \wedge b = \neg\Lambda, \end{cases}$$

If we ignore the distinction between 1 and Λ, we can say that $\diamond$ is just a special case of $\wedge$, in which the two flats have complementary ranks. If $a \diamond b = +1$, I will say that *a is positively oriented with respect to b*, or that *the pair a, b is positively oriented.*

The operation $a \diamond b$ is defined if and only if the flats have complementary ranks (that is, $\text{rank}(a) + \text{rank}(b) = \text{rank}(\Upsilon)$), and is $\mathbf{0}$ if and only if a and b have a common point. Observe also that $a \diamond b$ depends on the orientation of Υ, as well as on those of a and b.

As in any join, reversing the order of the arguments reverses the sign of $a \diamond b$ if and only if both have odd rank:

$$b \diamond a = (-1)^{\text{rank}(a)\,\text{rank}(b)}(a \diamond b)$$

Since in this case $\text{rank}(a)+\text{rank}(b) = n$, we conclude that the order of the arguments only matters if the space has even rank (odd dimension) and one of the operands has odd rank(even dimension). In three dimensions or less, the only cases where the order of a and b matters are two points on a line, and a point versus a plane in three-space.

2.1. Signed predicates

The $\diamond$ function is one of many sign-valued functions that are common in oriented projective geometry. The corresponding functions in unoriented geometry have only two outcomes, and are usually implemented as predicates, that is, procedures returning a boolean result. For example, the analogous of $\diamond$ would be the predicate that tests whether a given point lies on a given line.

In oriented projective geometry, it is generally better to implement a function like $\diamond$ as a procedure returning an integer value in $\{-1, 0, +1\}$. This procedure can be used both in two-way if statements, as in `if` $\texttt{Rel}(p, l) = 0$ `then` ..., and in three-branched `case` statements. Experience seems to show that when an algorithm of two-sided geometry needs to test a point against a line, more often than not it needs to take a different course of action for each of the three possible outcomes.

2.2. The two sides of an hyperplane

Let's now examine some special cases of $\diamond$ in more detail. If a is a point and h is a hyperplane, we say that *a is in the positive side of h* if and only if $a \diamond h = +1$. (To avoid ambiguity, I will not use the names "left side" and "right side" unless a is a point and b is a line of $\mathbf{T}_2$.) The order of the two operands matters if and only if the space has odd dimension; in that case, to correctly identify the positive side we must put the point on the left side of $\diamond$.

2.3. Two points on a line

In particular, on the two-sided line $\mathbf{T}_1$ the operation $p \diamond q$ tests whether the points p, q form a positive or a negative simplex of Υ, that is, whether q is ahead of or behind p on Υ_1. In the spherical model, this tests whether the shortest arc from p to q (in the spherical model) is counterclockwise ($p \diamond q = +1$) or clockwise ($p \diamond q = -1$). The test returns **0** if $p = q$ or $p = \neg q$. The positive side of a point (= hyperplane) q is therefore the half-line *ending* at q, i.e. the arc from $\neg q$ to q.

2.4. The two sides of a plane

A plane π divides $\mathbf{T}_3$ in two half-spaces. By definition, p is on the *positive side* of π if $p \vee \pi = \mathbf{T}_3$, and on the *negative side* if $p \vee \pi = \neg\mathbf{T}_3$. Let $(q; r; s)$ be a positive simplex of π; according to this definition, p is on the positive side of π if and only if $(p; q; r; s)$ is a positive tetrahedron of $\mathbf{T}_3$. See figure 3.

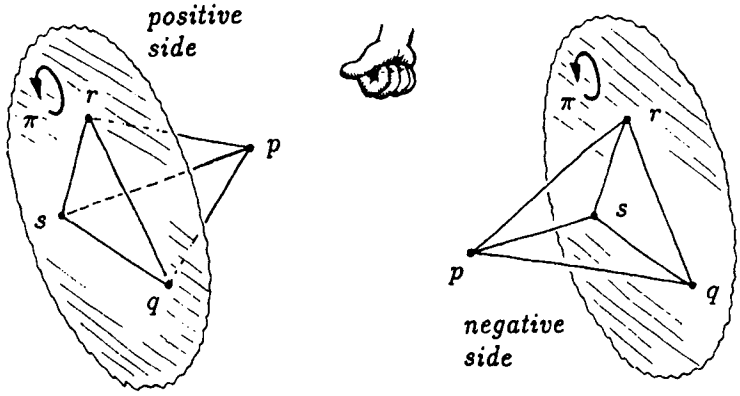

Figure 3. Testing a point against a plane.

Let's see what the two sides of a plane look like in the straight model. Recall that in the straight model of $\mathbf{T}_3$ a finite plane π is represented by two copies of some Euclidean plane of $\mathbf{R}^3$, one in each range. Let L^+ and R^+ be the two half-spaces of $\mathbf{R}^3$ determined by the front copy on the front range, and L^-, R^- their antipodes

on the back range. Then the two sides of π consist of $L^+ \cup R^-$ and $R^+ \cup L^-$. See figure 4.

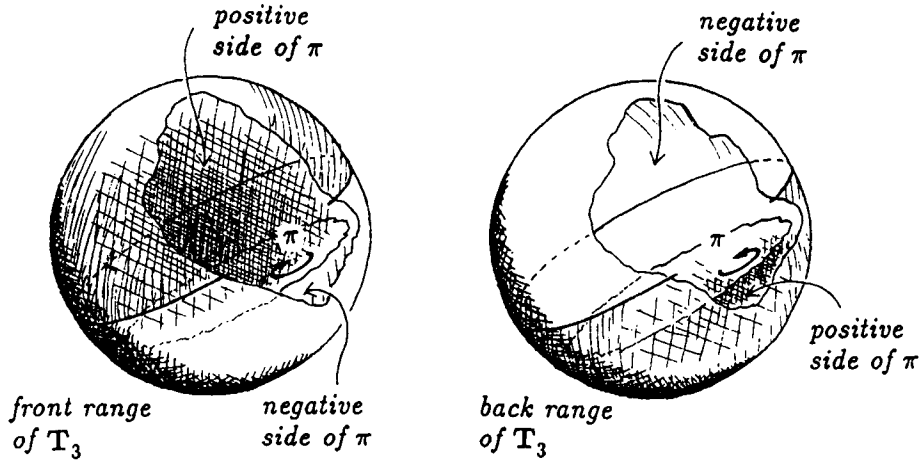

Figure 4. The two sides of a plane in $\mathbf{T}_3$.

Intuitively, the positive side of a plane π in $\mathbf{T}_3$ is the side from which the circular arrow of π seems to turn *clockwise*. Alternatively, we are in the positive side of π if, when we move towards any point q on π by the shortest route, the direction of travel $p \rightarrow q$ and the circular arrow of π at q are like the thumb and fingers of the right hand.

Observe that since planes and points have odd rank, $p \vee \pi = \neg(\pi \vee p)$. Therefore, the order of the join above is very important: to test whether a point is on the positive side of a plane, we join the point to the plane, not vice-versa.

2.5. The hyperplane at infinity

The set of all points at infinity of $\mathbf{T}_\nu$ (in the straight model) forms an unoriented hyperplane. I will denote by Ω_ν (or just Ω) that hyperplane, oriented so that the front origin O is on its positive side; that is $O \diamond \Omega = +1$.

In the case of $\mathbf{T}_3$, the orientation of Ω is what we would obtain by expanding the unit 2-sphere of the front range, oriented counterclockwise, to infinite radius. Therefore, for an observer at the origin of the front range, the "big circular arrow in the sky" will be turning *clockwise*.

2.6. Two lines in three-space

A less obvious use of $\diamond$ is in testing the relative orientation of two lines l, m in $\mathbf{T}_3$. The test $l \diamond m$ is positive if and only if a positive pair $(p; q)$ on l followed by a positive pair $(r; s)$ on m form a positive tetrahedron $(p; q; r; s)$ of $\mathbf{T}_3$. Intuitively, this means that m "turns around" l according to the right-handed rule. See figure 5. The test is zero if and only if the two lines intersect.

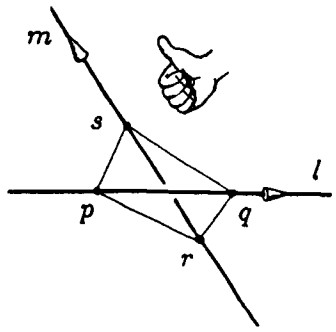

Figure 5. Two positively oriented lines of $\mathbf{T}_3$.

Note that l turns around m the same way that m turns around l. This is to be expected, since the join of two lines (rank 2) is commutative.

3. The separation theorem

An important axiom of Euclidean geometry states that if two points of the plane are distinct, they can be separated by a straight line. This axiom has a counterpart in oriented projective geometry: if two points p, q of $\mathbf{T}_2$ are distinct, there is a line l such that $p \diamond l \neq q \diamond l$. More generally, we have

Theorem 1. *Two flats a, b with the same rank are distinct if and only if there is some flat x in Υ with complementary rank such that $a \diamond x = -(b \diamond x)$.*

PROOF: The "if" is trivial, so let's prove the "only if" part. Let a, b be distinct flats with same rank k. Consider first the case where a and b are disjoint, and $a \vee b = \Upsilon$. That means $\text{rank}(\Upsilon) = 2k$. If $a = \Lambda$, then b must be $-\Lambda$, and $x = \Lambda$ will satisfy the theorem. Otherwise, let $(a^0; .. a^k)$ and $(b^0; .. b^k)$ be representative simplices of a and b. Construct the simplex $(x^1; .. x^\nu)$ where x^i (viewed as a vector of $\mathbf{R}^n$) is $\sigma_i a^i + b^i$, for some coefficients σ_i to be determined.

Now observe that the $2k \times 2k$ matrices M and N which relate the two simplices $(a^0; .. a^\kappa; x^0; .. x^\kappa)$ and $(b^0; .. b^\kappa; x^0; .. x^\kappa)$ to $(a^0; .. a^\kappa; b^0; .. b^\kappa)$ are

$$M = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ 0 & & 1 & & & \\ \sigma_0 & & 0 & 1 & & \\ & \ddots & & & \ddots & \\ 0 & & \sigma_\kappa & 0 & & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & & 0 & 1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 0 & 0 & & 1 \\ \sigma_0 & & 0 & 1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \sigma_\kappa & 0 & & 1 \end{pmatrix}$$

Since the determinant of M is $+1$, independently of the σ_i, we have $a \diamond x = +1$. On the other hand, the determinant of N is $(-1)^{k^2} \sigma_0 \cdots \sigma_\kappa$, which can be made -1 by a suitable choice of the σ_i. In that case we have $b \diamond x = -1$, and $a \diamond x = -(b \diamond x)$.

Now let a and b be arbitrary flats of Υ with same rank k. Let c be their intersection, arbitrarily oriented, and let u, v be right complements of c in a and b, so that $c \vee u = a$ and $c \vee v = b$. Obviously, we must have $u \neq v$, and $\text{rank}(u) = \text{rank}(v) = m$ for some $m \leq k$. Since v is disjoint from $a \cap b$ and contained in b, it is disjoint from a; therefore, $c \vee u \vee v$ is well-defined. Let d be such that $c \vee u \vee v \vee d = \Upsilon$. By the discussion above, there is a flat y of rank m in $u \vee v$ such that $u \vee y = u \vee v = \neg(v \vee y)$. Let $x = y \vee d$; we have

$$\begin{aligned} a \vee x &= c \vee u \vee v \vee d = c \vee u \vee v \vee d = \Upsilon, \\ b \vee x &= c \vee v \vee y \vee d = c \vee \neg(u \vee v) \vee d = \neg\Upsilon. \end{aligned} \tag{2}$$

Therefore, $a \diamond x = -(b \diamond x)$.

QED.

It follows immediately that

Theorem 2. *A flat a of $\mathbf{T}_\nu$ is uniquely characterized by the sign-valued function $x \mapsto a \diamond x$, where* $\text{rank}(x) + \text{rank}(a) = n$.

Note that this result holds even if $a \in \{\Lambda, \neg\Lambda, \Upsilon, \neg\Upsilon\}$, and can be extended to the case where a is $\mathbf{0}^k$, for all k.

4. The coefficients of a hyperplane

In the homogeneous model, a hyperplane h of $\mathbf{T}_\nu$ is represented by an oriented linear subspace H of $\mathbf{R}^n$ of dimension $n-1$. For any point $p=[x]$ of $\mathbf{T}_\nu$, the predicate $p \diamond h$ tests on which side of H the vector x lies. If y is any vector of $\mathbf{R}^n$ orthogonal to the space H and directed into its positive side, then testing $p \diamond h$ is equivalent to testing whether the projection of x onto the one-dimensional subspace generated by y has the same sign as y. In other words, $p \diamond h$ tests the sign of the dot product of x and y:

$$p \diamond h = \operatorname{sign}(x_0 y_0 + x_1 y_1 + \cdots + x_\nu y_\nu) \tag{3}$$

From this formula and from the separation theorem it follows that the hyperplane h is uniquely determined by the vector y. The coordinates of y are called the *homogeneous coefficients* of the hyperplane. I will use $\langle y \rangle$ to denote the hyperplane whose homogeneous coefficients are the coordinates of the vector y. As in the case of homogeneous coordinates of points, it is obvious from equation (3) and from the separation theorem that $\langle y \rangle = \langle z \rangle$ if and only if $y = \alpha z$ for some positive real number α.

It is often convenient to view y as a column vector $y = y(y^0; .. y^\nu)$, so that formula (3) can be written as a matrix product

$$\begin{aligned} [x] \diamond \langle y \rangle &= \operatorname{sign}(x_0 y^0 + \cdots + x_\nu y^\nu) \\ &= \operatorname{sign}\Big((x_0, .. x_\nu) \cdot \begin{pmatrix} y^0 \\ \vdots \\ y^\nu \end{pmatrix} \Big) \end{aligned} \tag{4}$$

Again, note that in spaces of odd dimension the order of the arguments of $\diamond$ is important, since

$$\langle y \rangle \diamond [x] \;=\; (-1)^\nu \, ([x] \diamond \langle y \rangle) \;=\; (-1)^\nu \operatorname{sign}(x_0 y^0 + \cdots + x_\nu y^\nu) \tag{5}$$

Chapter 8
Projective maps

The idea of projective maps grew out of the perspective rendering techniques developed by Renaissance artists. The perspective projection consists of extending a line from each point of the scene to the observer's eye, and marking the point where that line intersects the picture plane. See figure 1.

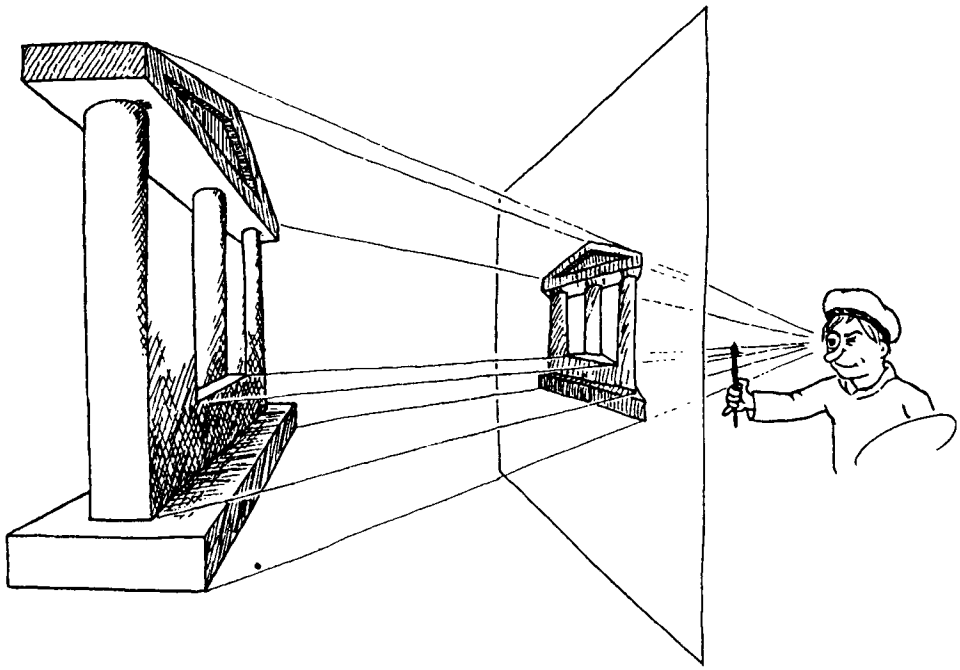

Figure 1. The perspective projection.

As figure 2 shows, even flat objects appear greatly distorted when viewed in perspective. The projection does not preserve many common geometric properties — angles, distances, areas (or their ratios), parallelism, perpendicularity, congruence, and so forth. Indeed, by moving the viewer and the projection screen appropriately, we can make the image of any convex quadrilateral on the floor to match any convex quadrilateral drawn on the screen. Informally, the two-dimensional projective

maps are all possible mappings from the floor plane to the screen plane that can be realized in this way.

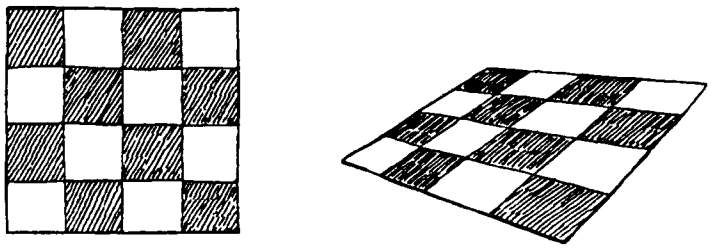

Figure 2. A chessboard, and a perspective view of it.

However, the perspective projection always takes straight lines to straight lines. Therefore, it preserves all geometric properties that can be defined in terms of incidence and collinearity. For example, any three collinear points or three concurrent lines will remain so in projection. This includes parallel lines, which are assumed to be concurrent at an infinity point. Among other things, this implies that no perspective view of a quadrilateral can be a circle or a pentagon.

In geometry texts this line-preserving property is usually taken as the definition of a *projective map* or *collineation* between two projective spaces. It can be shown that for spaces of dimension two or more, these maps correspond to linear maps in the analytic model. For technical and expository reasons, however, I will reverse this path, starting from an analytic definition of projective maps, and deriving from it their geometric properties.

1. Formal definition

Definition 1. Let S and T be two flats of $\mathbf{T}_\nu$, represented in the homogeneous model by two linear vector spaces U and V. A function M from the points of S to those of T is a *projective map* if it takes positive simplices of S to positive simplices of T, and there is a linear map M from U to V such that, for all $u \in U$,

$$M([u]) = [M(u)].$$

We can say that the linear map M of definition 1 *induces* the projective map M, and denote that by $[\![M]\!] = M$. Note that a projective map from S to T is also a projective map from $\neg S$ to $\neg T$, but not from S to $\neg T$. In other words, a projective map does not impose a global orientation on either its domain or its range, but establishes a rigid connection between the two. A projective map of a flat S to itself

is also called an *orientation-preserving map of* S, while a map from S to $\neg S$ is an *orientation-reversing* one.

To make formulas more readable, I will use xF as a synonym of $F(x)$, the image of an element x by a function F. I will also denote by FG the composition of functions F and G, applied in that order; that is, $xFG = (xF)G = G(F(x))$. I will use I_A (or just I) for the identity map on a set A, and $\overleftarrow{F}$ for the inverse of a (one-to-one) function F.

1.1. Equivalence of projective maps

Note that a given projective map may be induced by many distinct linear maps. The theorem below characterizes these maps:

Theorem 1. *Two linear maps K, L induce the same projective map if and only if $K = \alpha L$ for some $\alpha > 0$.*

PROOF: The "if" part is a trivial consequence of the definition. As for the "only if" part, assume K and L induce the same map M. Then K and L must have the same domain U and same range V (two linear subspaces of $\mathbf{R}^n$). The composite map $F = K\overleftarrow{L}$ must induce the identity projective map: for any $u \in U$, we must have $[uF] = [u]$.

In particular, let $(u^0; .. u^\kappa)$ be a basis of U. For any two distinct elements u, v of the basis we must have $[uF] = [u]$, $[vF] = [v]$, and $[(u+v)F] = [u+v]$. This means $uF = \alpha u$, $vF = \beta v$, and $(u+v)F = \gamma(u+v)$ for some $\alpha, \beta, \gamma > 0$. Since F is linear, we have also $(u+v)F = (uF) + (vF) = \alpha u + \beta v$, and therefore

$$\alpha u + \beta v = \gamma(u+v) = \gamma u + \gamma v$$

Since u and v are independent, we must have $\alpha = \beta$. Since this holds for all pairs u, v, there is a single $\alpha > 0$ such that $(u^i)F = \alpha u^i$ for all i. We conclude that F is a positive multiple $\alpha \cdot I_U$ of the identity map on U, and therefore

$$K = (K\overleftarrow{L})L = FL = \alpha \cdot I_U L = \alpha \cdot L.$$

QED.

2. Maps of $\mathbf{T}_\nu$ to itself

Of particular interest are the projective maps of $\mathbf{T}_\nu$ to itself, which are induced by linear maps of $\mathbf{R}^n$ to itself. By definition, any such map must preserve the sign of every ν-dimensional proper simplex, which means the inducing linear map must take positive bases to positive bases. As we know from linear algebra, this happens if and only if the coefficient matrix of the map has positive determinant. If the matrix has negative determinant, it induces an orientation-reversing map, which takes $\mathbf{T}_\nu$ to $\neg\mathbf{T}_\nu$. I will denote by $\mathcal{M}_\nu$ the set of all projective maps from $\mathbf{T}_\nu$ to itself.

2.1. Examples

As an example, consider the projective map of $\mathbf{T}_2$ induced by the linear transformation $(w,x,y) \mapsto (w,\ x+w,\ y+w/2)$ of $\mathbf{R}^3$. Figure 3 shows the effect of this map in the spherical model. In the straight model, this map is simply a translation of the front and back ranges by the vector $(1,1/2)$.

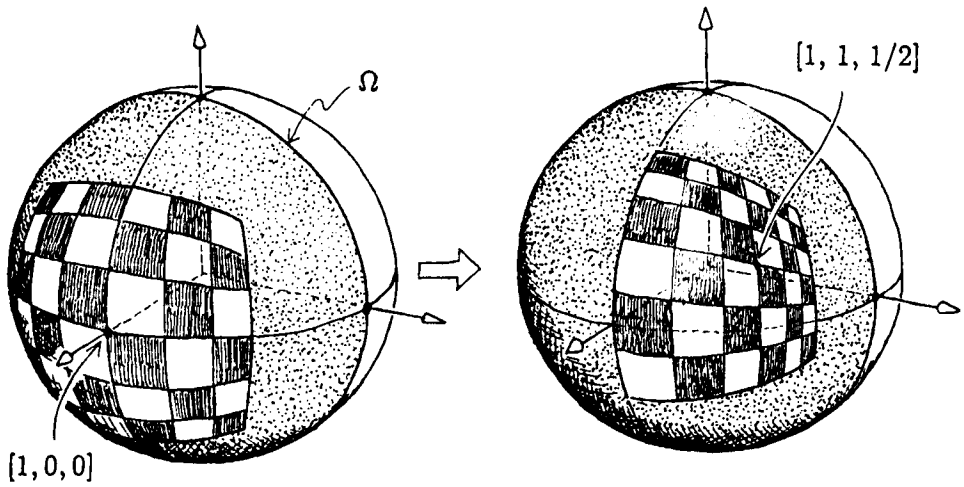

Figure 3. A translation.

Another example is the map $(w,x,y) \mapsto (10w,\ 4x-3y,\ 3x+4y)$. The induced projective map performs a rotation around the origin by an angle $\theta = \arctan(3/4)$, combined with a reduction by a factor of $\sqrt{3^2+4^2}/10 = 1/2$. See figure 4.

The examples above map the line at infinity Ω onto itself. It is easy to see that such maps are precisely those that correspond in the straight model to affine transformations of the front and back ranges. A projective map that doesn't belong to this class is the one induced by $(w,x,y) \mapsto (w-x,\ x,\ y)$. In the straight model,

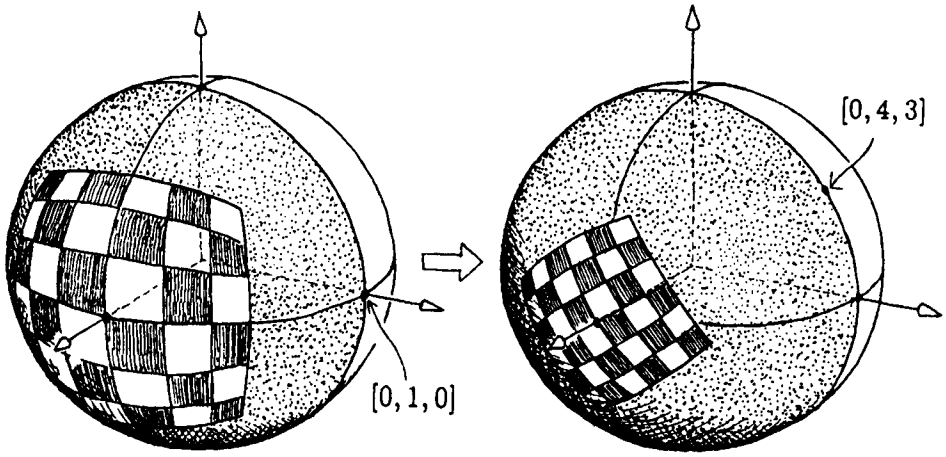

Figure 4. A similarity transformation.

this map brings the line Ω to the vertical line through $x = -1$. See figure 5.

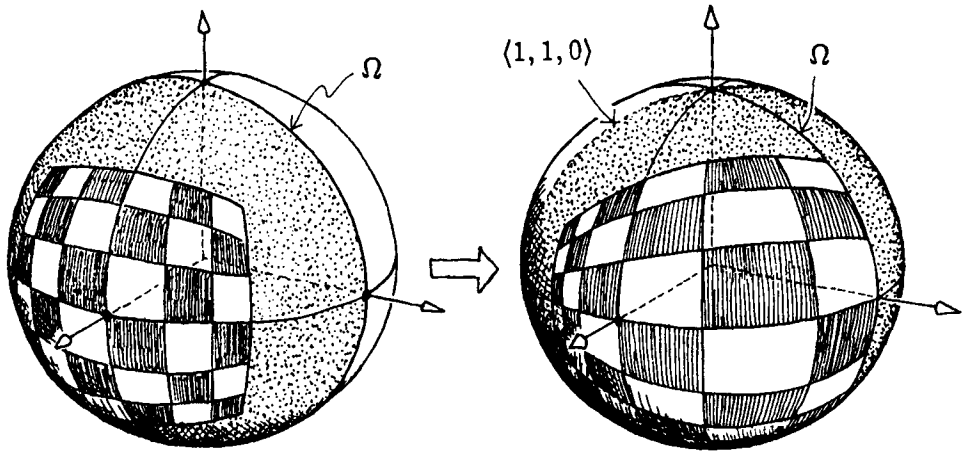

Figure 5. A perspective map.

Its effect can be described as a perspective projection between two copies of $\mathbf{T}_2$ properly positioned in $\mathbf{T}_3$. Note that parallel lines are mapped to lines that converge on the line $x = -1$. Note also that a portion of the back range is mapped onto the front range, and vice-versa.

3. Properties of projective maps

3.1. Group properties

Every projective map $M = [\![M]\!]$ is continuous; this follows from the continuity of M, and from the fact that $[u]$ is a continuous function of u. Any such M has a functional inverse $\overleftarrow{M} = [\![\overleftarrow{M}]\!]$, and is therefore one-to-one and onto. We have proved that

Theorem 2. *Every projective map from a flat S to a flat T is a topological homeomorphism between the point sets of S and T.*

Projective maps are closed under composition. The restriction of a projective map M to a flat subset Z of its domain is a projective map from Z to ZM. For any flat S, the identity function on the points of S is a projective map from S to itself. We conclude that

Theorem 3. *For any flat S, the projective maps of S to itself form a group under composition.*

3.2. Image of flats

Although a projective map $M = [\![M]\!]$ is defined as a mapping from points to points, it extends naturally to a mapping from flats to flats by the equations

$$\begin{aligned} M(\Lambda) &= \Lambda \\ M(\neg\Lambda) &= \neg\Lambda \\ M([u^0; .. u^\kappa]) &= [M(u^0); .. M(u^\kappa)] \end{aligned} \tag{1}$$

for any tuple $u^0; .. u^\kappa$ of linearly independent vectors in the domain U of M. In order for equation (1) to make sense, we have to show that the value of $M([u^0; .. u^\kappa])$ does not depend on the basis $u^0; .. u^\kappa$, but only on the linear space generated by it. In other words, we have to show that

$$[u^0; .. u^\kappa] = [v^0; .. v^\kappa] \;\Rightarrow\; [M(u^0); .. M(u^\kappa)] = [M(v^0); .. M(v^\kappa)] \tag{2}$$

The left-hand side of equation (2) says that the ordered bases $u^0; .. u^\kappa$ and $v^0; .. v^\kappa$ are equivalent, that is, there is a matrix of coefficients $A = (a^i_j \;:\; i, j \in 0..\kappa)$ with positive determinant such that $u^i = \sum_j a^i_j v^j$. Since M is linear, it follows that $M(u^i) = \sum_j a^i_j M(v^j)$, which implies the right-hand-side of equation (2). This shows definition 1 is not ambiguous. The requirement that M takes positive simplices of S

to positive simplices of T guarantees that the vectors $M(u^0);..M(u^\kappa)$ of equation (1) are independent

3.3. Meet and join

It follows immediately from (1) that, for every projective map M and any two flats a, b in its domain,

$$(\neg a)M = \neg(aM)$$

and

$$(a \vee b)M = (aM) \vee (bM)$$

Note that a projective map M from a flat S to a flat T takes by definition the universe of S to that of T (including orientation). Since the meet operation is defined in terms of join and the universe, we also have

$$(a \wedge_S b)M = (aM) \wedge_T (bM).$$

From these results we conclude that a projective map from S to T is an isomorphism between the projective structures of S and T (which includes all notions which can be defined in terms of meet and join).

4. The matrix of a map

A projective map M from $\mathbf{T}_\mu$ into $\mathbf{T}_\nu$ is induced by a linear map M of $\mathbf{R}^m$ into $\mathbf{R}^n$. Such a map can be represented by an $m \times n$ matrix of coefficients

$$\begin{pmatrix} m_0^0 & \cdots & m_\nu^0 \\ \vdots & \ddots & \vdots \\ m_0^\mu & \cdots & m_\nu^\mu \end{pmatrix}$$

with the convention that the image of a point $x = (x_0,..x_\nu)$ is

$$(xM)_j = \sum_i x_i m_j^i \tag{3}$$

If we view an element of $\mathbf{R}^n$ as a $1 \times n$ (i.e., row-like) matrix, we can express formula (3) as the matrix product

$$(x_0,..x_\nu)M = (x_0,..x_\nu) \cdot \begin{pmatrix} m_0^0 & \cdots & m_\nu^0 \\ \vdots & \ddots & \vdots \\ m_0^\nu & \cdots & m_\nu^\nu \end{pmatrix} \tag{4}$$

Of course, formulas (3) and (4) describe also the induced map M, since by definition $[x]M = [xM]$. In other words, to compute the image of a point we post-multiply its homogeneous coordinate vector by the coefficient matrix.[1]

From theorem 1 it follows that two projective maps from $\mathbf{T}_\mu$ into $\mathbf{T}_\nu$ are equal if and only if their matrices are positive multiples of each other. Therefore, we can identify a projective map M with the class of all matrices M that induce it. Accordingly, I will write

$$\left[\!\left[\begin{matrix} m_0^0 & \cdots & m_\nu^0 \\ \vdots & \ddots & \vdots \\ m_0^\mu & \cdots & m_\nu^\mu \end{matrix} \right]\!\right]$$

to denote the projective map generated by that matrix.

4.1. Examples

Here are some examples of maps from $\mathbf{T}_\nu$ to itself in matrix form and their effect (described in terms of the straight model):

$$\left[\!\left[\begin{matrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right]\!\right] \quad \text{Translation by } (x, y). \tag{5}$$

$$\left[\!\left[\begin{matrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{matrix} \right]\!\right] \quad \text{Rotation by angle } \theta \text{ around origin.} \tag{6}$$

$$\left[\!\left[\begin{matrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{matrix} \right]\!\right] \quad \text{Expansion by } \beta \text{ in the } x\text{-direction and by } \gamma \text{ in the } y\text{-direction.} \tag{7}$$

[1] Some authors prefer to view the elements of $\mathbf{R}^n$ as *column* vectors, and describe a linear map as *pre*-multiplication by the transpose of the matrix above. I adopted the present convention for consistency with the postfix notation xM and the left-to-right composition rule.

4.2. Image of a hyperplane

Geometric computations deal with hyperplanes almost as often as they do with points, so we need to derive the formulas for the homogeneous coefficients of the hyperplane hM given those of h.

First let's assume M is a positive (orientation-preserving) map. Such a map must preserve the relative orientation of points and hyperplanes, and therefore

$$p \diamond h = (pM) \diamond (hM) \qquad \text{for all points } p \text{ of } \mathbf{T}_\nu. \tag{8}$$

In fact, this property completely characterizes the hyperplane hM.

Now let $p = [u]$ and $h = \langle v \rangle$. If we view u as a row matrix, and v as a column one, then $p \diamond h$ is the sign of the matrix product $u \cdot v$ (which is a scalar). If M is the matrix of map M, and $\overleftarrow{M}$ its inverse, we have also the obvious identity

$$u \cdot v = (u \cdot M) \cdot (\overleftarrow{M} \cdot v) \tag{9}$$

The term $u \cdot M$ is a row vector, the coordinates of the point pM. The term $\overleftarrow{M} \cdot y$ is a column vector, the coefficients of some hyperplane g. Equation (9) says therefore

$$p \diamond h = (pM) \diamond g \qquad \text{for all } p \in \mathbf{T}_\nu.$$

Comparing this with equation (8), we conclude $g = hM$. That is, we obtain the coefficients of hM by the matrix product $\overleftarrow{M} \cdot v$.

This conclusion is correct only as long as the determinant $|M|$ of M's matrix is positive. If $|M|$ is negative, then in equation (8) we must compute $\diamond$ relative to the space $\neg\mathbf{T}_\nu$. In terms of the standard $\diamond$, we should write instead $p \diamond h = \neg((pM) \diamond (hM))$. On the other hand, equation (9) is not affected, so the end result is $hM = [\![-\overleftarrow{M} \cdot v]\!]$.

We can condense both cases into the single formula $\text{sign}(|M|) \cdot \overleftarrow{M} \cdot v$. Moreover, since positive factors are irrelevant, we can replace $\text{sign}(|M|) \cdot \overleftarrow{M}$ by just $|M| \cdot \overleftarrow{M}$. This is the so-called *adjoint* or *adjugate* of the linear map M. Therefore,

Theorem 4. *For any hyperplane $\langle v \rangle$ and any non-degenerate projective map $M = [\![M]\!]$, we have $\langle v \rangle M = \langle \overline{M} \cdot v \rangle$, where $\overline{M}$ is the adjoint linear map $|M| \cdot \overleftarrow{M}$ of M.*

Chapter 9
General two-sided spaces

The preceding chapters defined the canonical two-sided spaces $\mathbf{T}_\nu$ of arbitrary dimension ν; it is now time to define general two-sided spaces in a more abstract way. Among other things, this will allow us to view any flat of rank k in $\mathbf{T}_\nu$ as a copy of the canonical space $\mathbf{T}_\kappa$, and will prepare the ground for the discussion of duality in chapter 10.

Definition 1. The *canonical oriented projective space of dimension* ν is the quadruple $\mathbf{T}_\nu = (\mathcal{F}_\nu, \mathcal{M}_\nu, \vee_\nu, \wedge_\nu)$.

Definition 2. An *oriented projective space of dimension* ν is a quadruple $S = (\mathcal{F}_S, \mathcal{M}_S, \vee_S, \wedge_S)$, isomorphic to $\mathbf{T}_\nu = (\mathcal{F}_\nu, \mathcal{M}_\nu, \vee_\nu, \wedge_\nu)$.

By "isomorphic" we mean that (i) $\mathcal{M}_S$ must be a group of bijections from $\mathcal{F}_S$ to itself, (ii) $\vee_S$ and $\wedge_S$ must be binary operations on $\mathcal{F}_S$, and (iii) there must be a one-to-one mapping φ from $\mathcal{F}_S$ to $\mathcal{F}_\nu$ such that

$$\mathcal{M}_S = \{\, \overline{\varphi} M \varphi \;:\; M \in \mathcal{M}_\nu \,\}, \tag{1}$$

$$(a\varphi) \vee_\nu (b\varphi) = (a \vee_S b)\varphi, \text{ and} \tag{2}$$

$$(a\varphi) \wedge_\nu (b\varphi) = (a \wedge_S b)\varphi. \tag{3}$$

Informally, condition (1) says that $\mathcal{M}_S$ should act on the elements of $\mathcal{F}_S$ in the same way that $\mathcal{M}_\nu$ acts on the corresponding elements of $\mathcal{F}_\nu$. In particular, for every bijection $M \in \mathcal{M}_S$ there should be a projective map M^φ of $\mathbf{T}_\nu$ such that $M^\varphi = \overline{\varphi} M \varphi$, that is,

$$(sM)\varphi = (s\varphi)M^\varphi \quad \text{for all } s \in \mathcal{F}_S.$$

From what we saw in chapter 8, every projective map of $\mathbf{T}_\nu$ to itself is also an isomorphism of $\mathbf{T}_\nu$ to itself (an *automorphism* of $\mathbf{T}_\nu$). In fact, it is possible to prove that the only automorphisms of a projective space S are the projective maps of S to S.

1.1. Geometric operations

Additional geometric operations in a generic two-sided space S can be defined in terms of its join and meet operations. Equations (2) and (3) imply that the operations defined this way will coincide with the corresponding operations in $\mathbf{T}_\nu$, mapped through any isomorphism φ of S to $\mathbf{T}_\nu$. For example, the rank of any element $a \in \mathcal{F}_S$, defined as the number of points we have to join to get a, is the same as the rank in $\mathbf{T}_\nu$ of the flat $a\varphi$. The vacuum Λ_S of S (the neutral element of $\vee_S$) will always be $\Lambda\overline{\varphi}$, and the universe Υ_S of S (the neutral element of $\wedge_S$) will be $\Upsilon_\nu\overline{\varphi}$. Similarly, the operations of orientation reversal (join with Λ_S) and the relative orientation predicate (which compares the result of $\vee_S$ with Υ_S) satisfy

$$\neg_S\, a = (\neg\, a\varphi)\overline{\varphi} \tag{4}$$

$$a \diamond_S\, b = (a\varphi) \diamond (b\varphi) \tag{5}$$

where $\neg$, $\vee$, $\wedge$, and $\diamond$ denote the standard operations of $\mathbf{T}_\nu$.

2. Subspaces

We have already observed informally that a κ-dimensional flat of $\mathbf{T}_\nu$ looks pretty much like a copy of $\mathbf{T}_\kappa$. We can now state this more precisely:

Theorem 1. *Let*

s *be a* κ*-dimensional flat of* $\mathbf{T}_\nu$,
$\mathcal{F}_s$ *be the set of all flats of* $\mathbf{T}_\nu$ *contained in* s,
$\mathcal{M}_s$ *be the set of all projective maps from* s *to* s,
$\vee_s$ *be the join of* $\mathbf{T}_\nu$, *restricted to subflats of* s, *and*
$\wedge_s$ *be the meet operation relative to* s.

Then $(\mathcal{F}_s, \mathcal{M}_s, \vee_s, \wedge_s)$ *is an oriented projective space isomorphic to* $\mathbf{T}_\kappa$.

PROOF: Let $([u^0]; .. [u^\kappa])$ be any positive simplex of s. Recall this means $(u^0; .. u^\kappa)$ is a positive basis of the oriented linear subspace of $\mathbf{R}^n$ that represents s in the homogeneous model. Now consider the linear map E from $\mathbf{R}^k$ into $\mathbf{R}^n$ given by the matrix product

$$(x^0, .. x^\kappa) \mapsto (x^0, .. x^\kappa) \cdot \begin{pmatrix} u_0^0 & \cdots & \cdots & u_\nu^0 \\ \vdots & & & \vdots \\ u^\kappa & \cdots & \cdots & u_\nu^\kappa \end{pmatrix}$$

Let η be the map from $\mathbf{T}_\kappa$ into $\mathbf{T}_\nu$ defined by $[x]\eta = [xE]$ for all points $[x]$ of $\mathbf{T}_\kappa$, and extended to arbitrary flats of $\mathbf{T}_\kappa$ by the formula

$$(\Lambda_\kappa)\eta = \Lambda_\nu$$
$$(\neg\Lambda_\kappa)\eta = \neg\Lambda_\nu$$
$$[u^0;..u^\kappa]\eta = [E(u^0);..E(u^\kappa)]$$

These are essentially the same formulas we used for projective maps in chapter 8, and by the same arguments used there we can prove that η is well-defined. It is also easy to check that η is an isomorphism from $\mathbf{T}_\kappa$ to $(\mathcal{F}_s, \mathcal{M}_s, \vee_s, \wedge_s)$. QED.

I will call $(\mathcal{F}_s, \mathcal{M}_s, \vee_s, \wedge_s)$ the *subspace of* $\mathbf{T}_\nu$ *determined by* s. Not surprisingly, the orientation reversal operation in this subspace is that of $\mathbf{T}_\nu$, restricted to $\mathcal{F}_s$. The relative orientation test $a \diamond_s b$ checks whether $a \vee b$ is s or $\neg s$.

Note that the flats a and $\neg a$ determine different subspaces (they have different universes), even though $\mathcal{F}_a = \mathcal{F}_{\neg a}$ and $\mathcal{M}_a = \mathcal{M}_{\neg a}$. In particular, the subspace of $\mathbf{T}_\nu$ determined by $\neg\Upsilon_\nu$ is the space $\neg\mathbf{T}_\nu = (\mathcal{F}_\nu, \mathcal{M}_\nu, \vee, \bar{\wedge})$, where $x\bar{\wedge}y = \neg(x\wedge y)$. In this subspace the relative orientation predicate $\bar{\diamond}$ is such that $x\,\bar{\diamond}\,y = \neg(x \diamond y)$.

2.1. The canonical inclusion map

It is often useful to identify the space $\mathbf{T}_\kappa$ (for all $\kappa < \nu$) with the flat of $\mathbf{T}_\nu$ generated by the first m points of the standard simplex of $\mathbf{T}_\nu$. This flat consists of all points of $\mathbf{T}_\nu$ whose last $n - m$ coordinates are zero. The *canonical embedding* of $\mathbf{T}_\kappa$ in $\mathbf{T}_\nu$ is the function η that appends $n - k$ zeros to the coordinates of every point in $\mathbf{T}_\kappa$. Informally, η takes every point of $\mathbf{T}_\kappa$ to the "same" point of $\mathbf{T}_\nu$.

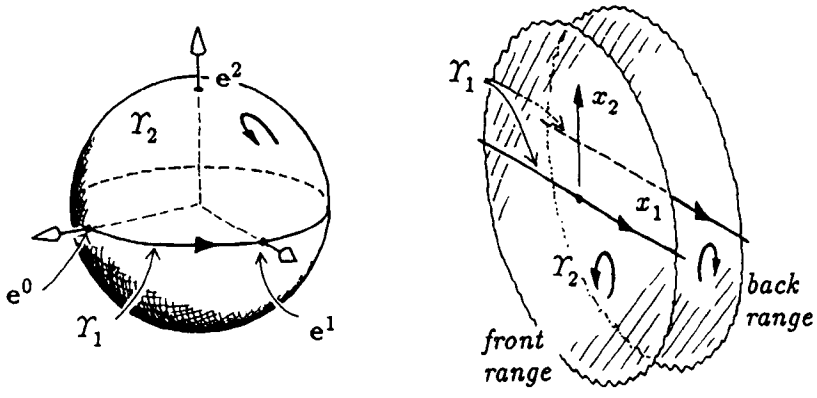

Figure 1. The canonical embedding of $\mathbf{T}_1$ in $\mathbf{T}_2$.

For example, the canonical embedding of $\mathbf{T}_1$ in $\mathbf{T}_2$ (in the spherical model) maps the unit circle $\mathbf{S}_1$ to the great circle determined on $\mathbf{S}_2$ by the plane of the first two coordinate axes. In the straight model, this great circle corresponds to the x-axis of the front and back ranges. See figure 1. Analytically, point $[w, x]$ of the two-sided line is mapped to point $[w, x, 0]$ of the two-sided plane.

2.2. Bundles

We can generalize further the notion of subspaces induced by flats as follows. Let s, t be two flats of $\mathbf{T}_\nu$, such that $s \supseteq t$. The *bundle* $\mathcal{F}_{s:t}$ determined by s and t consists of all flats that are contained in flat s and contain flat t. Let's denote by $\mathcal{M}_{s:t}$ the set of all projective maps from s to s that take the flat t to itself, with each map restricted to the flats that contain t. Let's also define the *join relative to* t of two flats a, b in this bundle, denoted by $\vee_t$, by the formula

$$(t \vee x) \vee_t (t \vee y) = t \vee x \vee y \tag{6}$$

We leave to the reader to verify that this definition is consistent, and that t is its neutral element. With these definitions, we have

Theorem 2. *For any two flats $s \supseteq t$, the quadruple $(\mathcal{F}_{s:t}, \mathcal{M}_{s:t}, \vee_t, \wedge_s)$ is a projective space of rank* $\operatorname{rank}(s) - \operatorname{rank}(t)$.

To prove this theorem, let c be any subflat of s that is a right complement of t in s (that is, $t \vee c = s$). Then consider the mapping φ from $\mathcal{F}_c$ to $\mathcal{F}_{s:t}$ defined by $x\varphi = t \vee x$ for every $x \in \mathcal{F}_c$. The inverse of this map takes every flat y in the bundle to the flat $y \wedge_s c$ of $\mathcal{F}_s$, where $\wedge_s$ is the meet operation relative to the flat s. It is easy to check that $\overline{\varphi}$ is an isomorphism from $(\mathcal{F}_{s:t}, \mathcal{M}_{s:t}, \vee_t, \wedge_s)$ to the subspace induced by c.

For example, consider the bundle $\mathcal{K}$ of all flats of $\mathbf{T}_2$ that contain the front origin $O = [1, 0, 0]$. See figure 2.

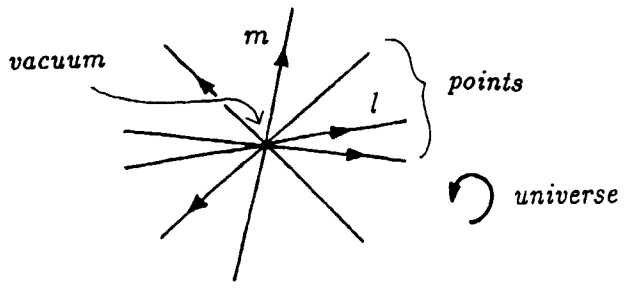

Figure 2. The bundle of all lines through the origin of $\mathbf{T}_2$.

The members of this set are the point $O = [1,0,0]$ and its antipode, all the lines passing through O (with coefficients of the form $\langle 0, X, Y\rangle$), and the two planes Υ_2, $\neg\Upsilon_2$.

The join operation $\vee_O$ of this bundle satisfies $O \vee_O x = x \vee_O O = x$ and $(\neg O) \vee_O x = x \vee_O (\neg O) = \neg x$ for all $x \in \mathcal{K}$. Also, for all lines $l, m \in \mathcal{K}$, the join $l \vee_O m$ is Υ_2 or $\neg\Upsilon_2$, according to whether the angle from l to m at O is positive or negative. Finally, for all other $a, b \in \mathcal{K}$, we have $a \vee_O b = \mathbf{0}^{\mathbf{rank}(a)+\mathbf{rank}(b)-1}$. The meet operation of this bundle is the standard meet operation of $\mathbf{T}_2$, and $\mathcal{M}_O$ is the set of all projective maps of $\mathbf{T}_2$, each restricted to the flats in $\mathcal{K}$. One of the many isomorphisms from $\mathbf{T}_1$ to this bundle is the map $\Lambda \mapsto O$, $\neg\Lambda \mapsto \neg O$, $[x, y] \mapsto \langle 0, x, y\rangle$, $\Upsilon_1 \mapsto \Upsilon_2$, and $\neg\Upsilon_1 \mapsto \neg\Upsilon_2$.

Other examples of bundles are the (one-dimensional) space of all planes of $\mathbf{T}_3$ containing a given line, or the (two-dimensional) space of all lines and planes containing a given point. Note that if we take $t = \Lambda$ we get simply the subspace determined by s. If we take $t = \neg\Lambda$, we get a space whose join operation $\bar{\vee}$ satisfies $x \,\bar{\vee}\, y = \neg(x \vee y)$.

We should note here that including both the projective maps and the operations of join and meet in the definition of a projective space is generally an overkill. More precisely, in spaces of dimension 2 or more a projective map can be defined as a bijection of $\mathcal{F}$ that commutes with the join operation. However, in one-dimensional spaces this requirement is too weak, and is satisfied by many functions that cannot be expressed as linear maps of the homogeneous coordinates. Accepting those functions as projective maps would be extremely inconvenient. For example, with the present definition every projective map defined on a line a can be extended to a projective map for any plane containing a. If projective maps were defined in terms of join, this would not be true.

On the other hand, the projective maps alone do not determine the join operation completely. Once we have join defined, we can define the meet operation in terms of the universe, or the universe in terms of meet. I have chosen the second alternative because it is more symmetric: as we shall see in chapter 10, the two operations are duals of each other.

Chapter 10
Duality

The reader may have observed that meet and join have very similar properties. In fact, most of the formulas we have seen so far occurred in pairs, where one member of the pair can be transformed into the other by exchanging meet with join, points with hyperplanes, rank with co-rank, vacuum with universe, and so forth. Compare for example the formulas

$$\begin{array}{cc} a \vee \Lambda = a & a \wedge \Upsilon = a \\ a \vee (\neg b) = \neg(a \vee b) & a \wedge (\neg b) = \neg(a \wedge b) \\ b \vee a = \neg^{rs}(a \vee b) & b \wedge a = \neg^{\bar{r}\bar{s}}(a \wedge b) \end{array}$$

where

$$\begin{array}{cc} r = \mathrm{rank}(a) & \bar{r} = \mathrm{corank}(a) \\ s = \mathrm{rank}(b) & \bar{s} = \mathrm{corank}(b) \end{array}$$

This phenomenon is known as the principle of *projective duality.* Its unoriented version is one of the most important ideas of classical projective geometry. In this chapter we will see that it holds in oriented geometry as well. As we may expect, the only difficulty is that we have to pay attention to the order of operands in expressions.

1. Duomorphisms

The duality between meet and join follows from a rather subtle result:

Theorem 1. *The quadruple* $\mathbf{T}^*_\nu = (\mathcal{F}, \mathcal{M}, \wedge, \vee)$ *is a projective space isomorphic to* $\mathbf{T}_\nu = (\mathcal{F}, \mathcal{M}, \vee, \wedge)$.

I will prove this theorem later on, by exhibiting an isomorphism η between the two spaces. Recall that an isomorphism η between those two quadruples must satisfy $\overline{\eta} M \eta \in \mathcal{M}$ for all $M \in \mathcal{M}$, and

$$(a\eta) \wedge (b\eta) = (a \vee b)\eta \text{ and}$$
$$(a\eta) \vee (b\eta) = (a \wedge b)\eta,$$

for all $a, b \in \mathcal{F}$. That is, join in the dual space $\mathbf{T}_\nu^*$ is the same as meet in $\mathbf{T}_\nu$, and vice-versa. From the properties of isomorphisms we proved in chapter 9, η must also satisfy

$$\begin{aligned} \Lambda\eta &= \Upsilon, \\ \Upsilon\eta &= \Lambda, \\ \neg(a\eta) &= (\neg a)\eta, \\ (a\eta) \diamond (b\eta) &= a \diamond b, \text{ and} \\ \text{rank}(a\eta) &= \text{corank}(a). \end{aligned}$$

The space $\mathbf{T}_\nu^*$ is called the *dual space* of $\mathbf{T}_\nu$, and any isomorphism between $\mathbf{T}_\nu$ and $\mathbf{T}_\nu^*$ is a *duomorphism* of $\mathbf{T}_\nu$.

1.1. Formal duality

The duomorphisms of $\mathbf{T}_\nu$ provide a solid foundation for the duality principle. Let $\mathcal{E}$ be a formula or assertion about the flats of $\mathbf{T}_\nu$, with no free variables, involving logical connectives and set operations, plus the symbols Λ, Υ, $\mathcal{F}$, $\mathcal{M}$, $\neg$,$\vee$, and $\wedge$, and any other operations that can be defined in terms of those. We construct the *formal dual* $\mathcal{E}^*$ of $\mathcal{E}$ by exchanging every occurrence of Λ with Υ, $\wedge$ with $\vee$, and recursively any derived concept with its formal dual. That includes swapping rank with co-rank, the word "point" with "hyperplane," the predicate $a \subseteq b$ (for flats) with $a \supseteq b$, and so on.

For example, the assertion "point x is on the segment pq" can be written as "$\text{rank}(p) = \text{rank}(q) = \text{rank}(x) = 1$ and $p \vee x = x \vee q = p \vee q \neq \mathbf{0}$." The dual of this is "$\text{corank}(p) = \text{corank}(q) = \text{corank}(x) = 1$ and $p \wedge x = x \wedge q = p \wedge q \neq \mathbf{0}$," which in $\mathbf{T}_2$ means the line x is concurrent with p and q, and its direction lies in the shorter angle between the directions of p and q.

As another example, consider predicate $a \diamond b = +1$. To construct its dual, we first rewrite it in terms of join, which gives $a \vee b = +\Upsilon$. The formal dual of this is $a \wedge b = +\Lambda$. By the definition of $\wedge$, this is the same as $a \vee b = +\Upsilon$. We conclude that $\diamond$ is its own dual. One can easily verify that the same is true of the orientation reversal operation $(\neg)$.

Meta-theorem 2. *If $\mathcal{T}$ is a theorem of oriented projective geometry, then its formal dual $\mathcal{T}^*$ is also a theorem.*

This is a straightforward exercise in formal logic, which we will not prove here since it falls somewhat outside the scope of this work.

2. The polar complement

It is now time to prove theorem 1. To do so I will exhibit one particular duomorphism for $\mathbf{T}_\nu$, the *polar complement* function.

2.1. Polar flats

I will say that two points of $\mathbf{T}_\nu$ are *polar* if they are represented by orthogonal vectors in the spherical model. In general, two flats a, b are *polar*, denoted by $a \perp b$, if every point on one of them is polar to every point on the other.

2.2. Polar complement

Recall that, according to theorem 5:1, for every flat a in Υ there is some *complementary flat* b such that $a \vee b = \Upsilon$ (and therefore $a \wedge b = \Lambda$, and $a \diamond b = +1$). In general, the flat b is not unique; for example, the complement of a point a in $\mathbf{T}_2$ can be any line leaving a on the left side. However, we can use the polarity predicate above to make the complement unique:

Definition 1. The *right polar complement* of a flat a is the flat $a^\vdash$ such that

$$\begin{aligned} a &\perp a^\vdash \\ a \diamond a^\vdash &= +1 \end{aligned} \tag{1}$$

It is not hard to see that $a^\vdash$ always exists and is unique, is a continuous function of a, is disjoint from a, and satisfies $\text{rank}(a^\vdash) = \text{rank}(\Upsilon) - \text{rank}(a) = \text{corank}(a)$. Symmetrically the *left polar complement* $\dashv$ is defined by

$$\begin{aligned} a^\dashv &\perp a \\ a^\dashv \diamond a &= +1 \end{aligned} \tag{2}$$

As usual, it is convenient to define also $(\mathbf{0}^k)^\vdash = (\mathbf{0}^k)^\dashv = \mathbf{0}^{n-k}$ for all k. The names "right complement" and "left complement" refer to the order of a, $a^\vdash$, and $a^\dashv$ in formulas (1–2), and not to the relative positions of those flats in $\mathbf{T}_\nu$. To get $+1$ we must write the right complement on the right side of the "$\diamond$", or the left complement on the left side. The symbols $\vdash$ and $\dashv$ were chosen to make this rule easier to remember.

It follows immediately from the definition that

$$(a^\vdash)^\dashv = a = (a^\dashv)^\vdash$$

That is, $\vdash$ and $\dashv$ are inverses of each other. We also have

$$(\neg a) \vee (\neg(a^{\vdash})) = a \vee a^{\vdash} = \Upsilon$$
$$(\neg a) \vee (\neg(a^{\dashv})) = a \vee a^{\dashv} = \Upsilon$$

which implies

$$(\neg a)^{\vdash} = \neg(a^{\vdash}) \qquad (\neg a)^{\dashv} = \neg(a^{\dashv}).$$

If $rank(a) = r$ and $\text{rank}(\Upsilon) = n$, it is easy to see that

$$a^{\vdash} \diamond a = \neg^{r(n-r)}(a \diamond a^{\vdash}),$$

which implies

$$a^{\vdash} = \neg^{r(n-r)} a^{\dashv} \tag{3}$$

and

$$(a^{\vdash})^{\vdash} = (a^{\dashv})^{\dashv} = \neg^{r(n-r)} a.$$

Observe that if n is odd, then for any r the product $r(n-r)$ is even. It follows that in spaces of odd rank (even dimension), $\dashv$ and $\vdash$ are the same function. This is the case, for example, in the plane $\mathbf{T}_2$. To emphasize this fact, I will in those spaces use the same symbol $a^{\perp}$ for both $a^{\vdash}$ and $a^{\dashv}$. On the other hand, in spaces of even rank (odd dimension), like $\mathbf{T}_3$, we have $a^{\vdash} = a^{\dashv}$ or $a^{\vdash} = \neg a^{\dashv}$, depending on whether the rank of a is even or odd. So, for example, the left and right complements of a point (rank 1) or a plane (rank 3) are opposite, whereas those of a line are the same.

2.3. Polar complements in the two-sided plane

Let's consider some examples. In the two-sided plane the two polar complements are the same function, $a^{\vdash} = a^{\vdash} = a^{\perp}$. In the spherical model, this function takes every oriented great circle of $\mathbf{S}_2$ to the apex of its left hemisphere, and, conversely, every point of $\mathbf{S}_2$ to the oriented great circle whose left hemisphere has that point at the apex. See figure 1(a). In the straight model, the image $l^{\perp}$ of a finite line l passing at distance $d > 0$ from the origin O is the point p such that the vector Op is perpendicular to l, directed away from l, and with length $1/d$. See figure 1(b). The point $l^{\perp}$ will be on the front range if and only if the line is directed counterclockwise

as seen from the front origin, i.e. if the origin lies on the positive side of l.

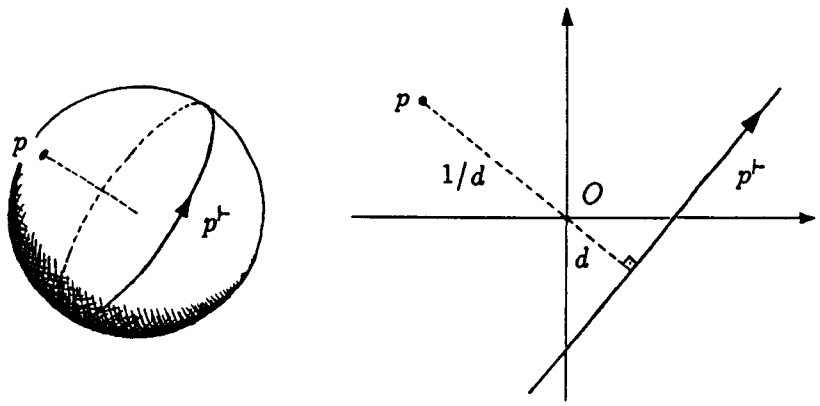

Figure 1. Polar complement in $\mathbf{T}_2$.

If the line l passes through the origin (i.e., $d = 0$), then $l^\perp$ is the point at infinity in the direction 90° counterclockwise from that of l. Conversely, if $l = \Omega$, then $l^\perp = O$ (the origin of the front range), and if $l = \neg\Omega$ then $l^\perp = \neg O$ (the origin of the back range). Observe how in all cases these definitions put $l^\perp$ on the left (positive) side of l.

2.4. Polar complements in three-space

Let us now consider the case of a point p in three-dimensional space $\mathbf{T}_3$. If p is on the front range and at distance $d > 0$ from the origin O, its polar complement $p^\perp$ is a plane perpendicular to the vector Op and at distance $1/d$ from O in the direction opposite to that of p. See figure 2.

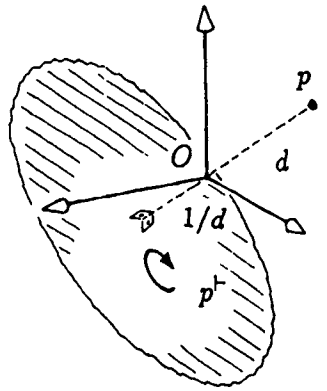

Figure 2. The polar complement of a point in $\mathbf{T}_3$.

If $p = O$ its polar complement is one of the two planes at infinity. If p is on the back range, the same applies with $\neg O$ substituted for O. If p is infinite, then the plane passes through O and is perpendicular to the direction from O towards p. Since $p \diamond p^{\vdash} = +1$, $p^{\vdash}$ is oriented so that p is on its positive side; that is, the circular arrow of $p^{\vdash}$ turns *clockwise* as seen from p. In particular, $O^{\vdash} = \Omega$. Since p has odd rank (1) and odd co-rank (3), we have $p^{\vdash} = \neg p^{\dashv}$.

Consider now a finite line l of $\mathbf{T}_3$ whose point p closest to the origin is at distance $d > 0$ from it. The right polar complement $l^{\vdash}$ is a line skew to but perpendicular to l. The point q of $l^{\vdash}$ that is closest to O is at distance $1/d$ from O, in the direction opposite to Op. See figure 3.

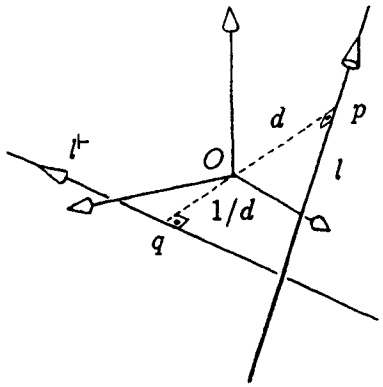

Figure 3. The polar complement of a line of $\mathbf{T}_3$.

If l is at infinity, $l^{\vdash}$ passes through the origin and is perpendicular to all planes containing l. The orientation of $l^{\vdash}$ is given by the right-hand rule. Since lines have even rank, their left and right polar complements coincide: $l^{\vdash\vdash} = l^{\dashv\dashv} = l$.

2.5. Polar complement in the analytic model

In the homogeneous model, right polar complement of a point $[x]$ is a hyperplane of $\mathbf{T}_\nu$, represented by a $(n-1)$-dimensional linear subspace of $\mathbf{R}^n$ that is orthogonal to the vector x. The homogeneous coefficients of that hyperplane are the coordinates of a properly oriented vector orthogonal to that linear subspace, that is, either x or $-x$ (or any multiple thereof). In other words, we have $[x] = \langle \sigma x^{\mathrm{tr}} \rangle$, where x^{tr} is x viewed as a column vector, and $\sigma \in \{+1, -1\}$. We can figure out the correct sign from the defining equation $a \diamond a^{\vdash} = +1$ and $[x] \diamond \langle z \rangle = \mathrm{sign}(x_0 z^0 + \cdots + x_\nu z^\nu)$, which imply

$$[x]^{\vdash} = \langle x^{\mathrm{tr}} \rangle.$$

It follows immediately that

$$\langle z \rangle^{\vdash} = \neg^{\nu} [z^{\mathrm{tr}}] \tag{4}$$

and, symmetrically,

$$[x]^{\dashv} = \neg^{\nu} \langle x^{\mathrm{tr}} \rangle \tag{5}$$

$$\langle z \rangle^{\dashv} = [z^{\mathrm{tr}}]. \tag{6}$$

3. Polar complements as duomorphisms

Let's then prove that $\vdash$ is indeed a duomorphism. Most of the proof is contained in the next two lemmas:

Lemma 3. *For any projective map M of $\mathbf{T}_{\nu}$, the function $\overleftarrow{\vdash} M \vdash = \dashv M \vdash$ is a projective map of $\mathbf{T}_{\nu}$.*

PROOF: Let M be a linear map of $\mathbf{R}^n$ that induces M. Equation (5) says that, for any point $[x]$ of $\mathbf{T}_{\nu}$, $[x]^{\dashv} = \neg^{\nu}\langle x^{\mathrm{tr}} \rangle$. From theorem 4 of chapter 10 we know that

$$\langle x^{\mathrm{tr}} \rangle M = \langle \bar{M} \cdot x^{\mathrm{tr}} \rangle,$$

where $\bar{M} = |M| \cdot \overleftarrow{M}$ is the adjoint of the matrix of M. Finally, from equation (4) we get

$$\langle \bar{M} \cdot x^{\mathrm{tr}} \rangle^{\vdash} = \neg^{\nu} [(M' \cdot x^{\mathrm{tr}})^{\mathrm{tr}}].$$

Putting all pieces together, we get

$$[x] \dashv M \vdash = [x \cdot \bar{M}^{\mathrm{tr}}],$$

that is, $\dashv M \vdash$ is the projective map $[\![\bar{M}^{\mathrm{tr}}]\!]$, whose matrix is the *untransposed* cofactor matrix of M.

QED.

Lemma 4. *The polar complement functions satisfy*

$$(a \vee b)^{\vdash} = a^{\vdash} \wedge b^{\vdash} \qquad (a \vee b)^{\dashv} = a^{\dashv} \wedge b^{\dashv}$$

$$(a \wedge b)^{\vdash} = a^{\vdash} \vee b^{\vdash} \qquad (a \wedge b)^{\dashv} = a^{\dashv} \vee b^{\dashv}$$

for all flats (or null objects) a, b.

PROOF: Let's prove first $(a \vee b)^{\vdash} = a^{\vdash} \wedge b^{\vdash}$. Let $c = (a \vee b)^{\vdash}$, $r = \mathrm{rank}(a)$, and $s = \mathrm{rank}(b)$, so that $\mathrm{rank}(c) = n - r - s \geq 0$ (Recall that for the join to be defined, we must have $r + s \leq n$). From the definition of $\vdash$, we get

$$(a \vee b) \diamond (a \vee b)^{\vdash} = +1,$$

that is,

$$a \vee b \vee c = \Upsilon, \tag{7}$$

which implies

$$a^{\vdash} = b \vee c. \tag{8}$$

Also, from (7) and the commutativity laws of join we have

$$b \vee c \vee a = \neg^{r(n-r)}\Upsilon, \tag{9}$$

which implies

$$b^{\vdash} = \neg^{r(n-r)}(c \vee a). \tag{10}$$

From this and (8), we get

$$\begin{aligned} a^{\vdash} \wedge b^{\vdash} &= \big(b \vee c\big) \wedge \big(\neg^{r(n-r)}(c \vee a)\big) \\ &= \neg^{r(n-r)}\,\big((b \vee c) \wedge (c \vee a)\big). \end{aligned}$$

From equation (9) and the definition of $\wedge$, we have

$$(b \vee c) \wedge (c \vee a) = \neg^{r(n-r)}c,$$

and therefore

$$a^{\vdash} \wedge b^{\vdash} = \big(\neg^{r(n-r)}\big)^2 c = (a \vee b)^{\vdash}, \tag{11}$$

as we proposed to show. As for the left complement, from equation (3) we get

$$\begin{aligned} a^{\dashv} &= \neg^{r(n-r)}\, a^{\vdash}, \\ b^{\dashv} &= \neg^{s(n-s)}\, b^{\vdash}, \\ (a \vee b)^{\dashv} &= \neg^{(r+s)(n-r-s)}\, (a \vee b)^{\vdash}, \end{aligned}$$

which plugged into equation (11) gives

$$\begin{aligned} (a \vee b)^{\dashv} &= \neg^{(r+s)(n-r-s)+r(n-r)+s(n-s)}\, (a^{\dashv} \wedge b^{\dashv}) \\ &= \neg^{2rn+2sn-2r^2-2rs-2s^2}\, (a^{\dashv} \wedge b^{\dashv}) \\ &= a^{\dashv} \wedge b^{\dashv}. \end{aligned} \tag{12}$$

Applying $\dashv$ to both sides of (11), and doing the variable substitutions $a^{\vdash} \mapsto a$, $b^{\vdash} \mapsto b$ (which are valid, since $\dashv$ and $\vdash$ are one-to-one and onto) we get

$$(a \wedge b)^{\dashv} = a^{\dashv} \vee b^{\dashv}.$$

In the same way, from equation (12) we get $(a \wedge b)^{\vdash} = a^{\vdash} \vee b^{\vdash}$. This concludes the proof.

QED.

The main result of this chapter then follows trivially from lemmas 3 and 4:

Theorem 5. *The function* $\vdash$ *is an isomorphism between the projective spaces* $(\mathcal{F}, \mathcal{M}, \wedge, \vee)$ *and* $(\mathcal{F}, \mathcal{M}, \vee, \wedge)$.

To conclude this section, observe that

$$\overleftarrow{\dashv} M \dashv = (\neg^{\nu} \overleftarrow{\vdash}) M (\neg^{\nu} \vdash) = \overleftarrow{\vdash} M \vdash$$

which shows $\dashv$ too is a duomorphism of $\mathbf{T}_{\nu}$.

4. Relative polar complements

We can generalize the definition of polar complement to arbitrary subspaces of $\mathbf{T}_{\nu}$ by using an arbitrary flat f in lieu of the universe. That is, for any flat $a \subseteq f$ we define the *right polar complement of* a *relative to* f as the flat $a \rceil f$ satisfying

$$a \perp (a \rceil f)$$
$$a \vee (a \rceil f) = f.$$

Symmetrically, the *left polar complement of* a *relative to* f satisfies

$$(f \lceil a) \perp a$$
$$(f \lceil a) \vee a = f.$$

In particular, $a^{\vdash} = a \rceil \Upsilon$, $a^{\dashv} = \Upsilon \lceil a$. Conversely, we have

$$a \rceil f = f \wedge a^{\vdash}$$
$$f \lceil a = a^{\dashv} \wedge f.$$

These are special cases of the following theorem:

Theorem 6. *For any flats* $a \subseteq f \subseteq g$,

$$a \rceil f = f \wedge (a \rceil g)$$
$$f \lceil a = (g \lceil a) \wedge f.$$

PROOF: First, observe that $f \wedge (a \rceil g))$ is contained in f and is polar to a (since the latter is true of $a \rceil g$). From $a \vee (a \rceil f) = f$ we get $a \vee (a \rceil f) \vee (f \rceil g) = \Upsilon$, hence

$$(a \rceil f) \vee (f \rceil g) = (a \rceil g).$$

Therefore

$$f \wedge (a \rceil g) = [a \vee (a \rceil f)] \wedge [(a \rceil f) \vee (f \rceil g)]$$
$$= a \rceil f.$$

QED.

Lemma 4 generalizes to relative complements through the equations

$$(a \vee b) \rceil f = (a \rceil f) \wedge (b \rceil f)$$
$$f \lceil (a \vee b) = (f \lceil a) \wedge (f \lceil b)$$
$$(a \wedge b) \rceil f = (a \rceil f) \vee (b \rceil f)$$
$$f \lceil (a \wedge b) = (f \lceil a) \vee (f \lceil b).$$

In general, there are no simple formulas for $a \rceil (f \vee g)$, $a \rceil (f \wedge g)$, or their $\lceil$ analogs.

5. General duomorphisms

In general, a *duomorphism* is an isomorphism between a two-sided space $S = (\mathcal{F}_S, \mathcal{M}_S, \vee_S, \wedge_S)$ and its *dual space* $S^* = (\mathcal{F}_S, \mathcal{M}_S, \wedge_S, \vee_S)$.

It is easy to check that the composition of a duomorphism η and a projective map (in either order) is also a duomorphism. In particular, the composition of the polar complement $\dashv$ and any projective map of $\mathbf{T}_\nu$ to itself is a duomorphism of $\mathbf{T}_\nu$.

Conversely, if η and φ are duomorphisms from S to S^*, then the composition $\eta\overleftarrow{\varphi}$ is obviously an isomorphism of S to itself, and therefore a projective map of S. It follows that

Theorem 7. *Every duomorphism of a space S can be written as the product of a fixed duomorphism of S and some projective map of S.*

In fact, if $\text{rank}(S) = m$, we can always write a duomorphism η of S as $\eta = M \vdash \overleftarrow{N}$, where M and N are projective maps from S to $\mathbf{T}_\mu$, and $\vdash$ is the right polar complement in $\mathbf{T}_\mu$. Actually, we can choose one of M and N arbitrarily, with the other being a function of η and the chosen map.

In particular, any duomorphism of $\mathbf{T}_\nu$ is the composition of the right polar complement $\vdash$ and a suitable projective map of $\mathbf{T}_\nu$ (or vice-versa). For example, the left polar complement $\dashv$ of $\mathbf{T}_\nu$ is the composition of $\vdash$ and the projective map

$$x \mapsto \neg^{n-1} x \quad \text{for every point } x.$$

Note how this map is the identity for spaces of even dimension (meaning $\dashv$ and $\vdash$ are the same function), and is the antipodal map $x \mapsto \neg x$ for spaces of odd dimension.

6. The power of duality

Duality is an extremely powerful tool. For one thing, it greatly reduces the number of theorems that have to be proved, since every proof automatically establishes the correctness of a theorem and its dual. Moreover, we can choose among the two theorems the one whose proof is easier to visualize, so we may end up doing much less than half the work.

Duality is equally valuable from a computational point of view, since it allows every geometrical algorithm to do the work of two. Thus, a subroutine that computes $a \vee b$ can be used to compute $a \wedge b$, by the formula $(a^{\vdash} \vee b^{\vdash})^{\dashv}$. As we shall see, with the proper representation $\dashv$ and $\vdash$ can be computed at negligible cost. Duality thus may cut the size of a geometric library (and of its documentation) by almost one half. Similar savings apply to higher-level algorithms; for example, an algorithm that computes the convex hull of n points can also be used to find the intersection of n half-spaces.

Chapter 11
Generalized projective maps

One-to-one projective maps are important because they are exactly the projective isomorphisms, i.e. the maps that preserve straight lines and orientations, and therefore join, meet, and all derived concepts. There is however a larger class of maps between flats that are not one-to-one, but still preserve those objects and operations to some extent.

1. Projective functions

One way to generalize the notion of projective map given by definition 8:1 is to drop the requirement that the linear map M be one-to-one and onto. The functions induced by such maps will be undefined for some points, but will still satisfy some of the properties of projective maps. I will call such functions *projective functions.*

More precisely, let S and T be two flats of $\mathbf{T}_\nu$, and U and V their homogeneous models. Let M be an arbitrary linear map from U into V, and consider the function $M = [\![M]\!]$ from the points of S into those of T that is induced by M according to

$$[x]M = [xM] \quad \text{for all } x \in \mathbf{R}^n.$$

As in the case of non-degenerate maps, two linear maps M, N induce the same projective function if and only if $M = \alpha N$ for some positive real α.

As we know from linear algebra, the map M is many-to-one if and only if it takes some non-zero vector to $(0,..0)$. In this case the induced function M will take some valid point of S to the indeterminate object $\mathbf{0}$. The set of such points is a flat subset of S, corresponding to the non-zero vectors in the null space (kernel) of the map M. By analogy, I will call that subset of S the *null space* of M, and denote it by $Null(M)$.

The natural extension of M to flats of S,

$$\begin{aligned} \Lambda M &= \Lambda \\ (\neg\Lambda)M &= \neg\Lambda \\ [u^0;..u^\kappa]M &= [(u^0)M;..(u^\kappa)M] \end{aligned}$$

is similarly incomplete. If the linear map M is not one-to-one, then the images of k independent vectors $u^0; .. u^\kappa$ from U may not be independent vectors of V. For the images to be independent, the linear subspace $<u^0; .. u^\kappa>$ or U must be disjoint from the null space of M; that is, the flat of S spanned by the simplex $([u^0]; .. [u^\kappa])$ must be disjoint from the flat set $Null(M)$.

In general, the image of a flat a by a projective function M is well-defined if and only if a (as a set of points) is disjoint from the null space of M. If that is not the case, the image of a is best defined as the indeterminate object $\mathbf{0}$ with the same rank as a. (We could define aM in this case as the unoriented flat set $\{ xM \ : \ x \in a \}$, which has rank strictly less than that of a. However, this would not be of much help in practice, and would make many other formulas and theorems needlessly complicated. This question is closely related to that of assigning meaning to $a \vee b$ when a and b are not disjoint, and most arguments relevant to the latter apply also to the former).

For a projective function that is not a map, the requirement that M preserves the sign of every representative simplex in its domain is no longer meaningful. If the inducing linear map M is not onto, then the image of a maximum-rank simplex of S has less than maximum rank in T. Conversely, if the map M is many-to-one, then any maximum-rank simplex of S will be mapped to a degenerate one.

1.1. The perspective projection

An important example of a projective function is the perspective projection of $\mathbf{T}_3$ that we mentioned before. Let π is the plane of the image, and p the position of the observer. The perspective projection will take an arbitrary point x on the object being rendered to the point

$$F(x) = (p \vee x) \wedge \pi \tag{1}$$

of the image plane. See figure 1. This projection is clearly not one-to-one, since it maps all the points in the ray ox to the same point of π.

Formula (1) is well defined as long as the point p is not on the plane π, and x is neither p nor $\neg p$. According to our previous conventions, we should let $F(p)$ and $F(\neg p)$ be the indeterminate object $\mathbf{0}$. When p is on π, then formula (1) specifies a degenerate map that takes every point x to $\neg p$, p, or $\mathbf{0}$, depending on whether $x \diamond \pi$ is positive, negative, or zero. In the sequel, we will assume this is not the case.

Formula (1) can be used also to give the perspective projection of an arbitrary flat x of $\mathbf{T}_3$. From the formula it is obvious that $F(x)$ is well defined as long as the flat x is not incident to the point p. For example, the projection of a line of $\mathbf{T}_3$ is a well-defined line of π, unless the line is seen end-on. When the flat x is incident to p, our rules define $F(x)$ as the undefined object $\mathbf{0}$ with same rank as x.

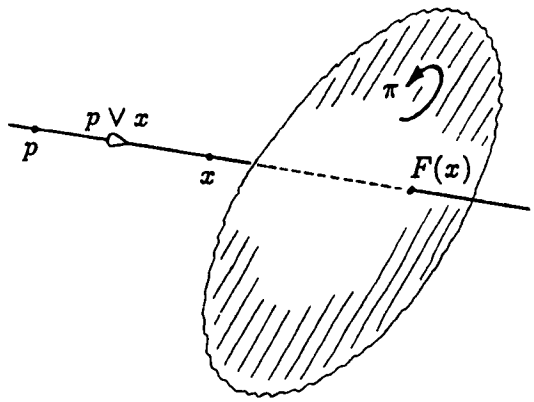

Figure 1. Perspective projection.

For perspective rendering it is convenient to orient π so that p is on its positive side. This implies $F(x) = x$ for all x on π; that is, F is idempotent, as is usually expected of a projection map. When p is on the negative side of π we obviously have $F(x) = \neg x$ for all flats x contained in π.

In typical perspective rendering applications π is a proper plane (not contained in Ω), and p is a point on the front range. Note that all points on the plane parallel to π and passing through p will project to a point at infinity on π. In the Cartesian framework those points must be handled as special cases, but of course in the projective framework they are just like any other point.

1.2. General and polar projections

Note that formula (1) can be extended to use any pair of complementary flats in lieu of p and π. That is, if a, b are subflats of a flat S with $a \vee b = S$, we define the *projection of S onto b from a* as the mapping

$$F(x) = (a \vee x) \wedge b$$

where the meet is computed relative to S. For example, if a and b are two skew lines in $\mathbf{T}_3$, F will be a map taking every point of $\mathbf{T}_3$ that is not on line a to a point on line b. The points of $\mathbf{T}_3$ with the same image y are precisely those on the plane $a \vee y$, except for those on a itself (which are mapped to $\mathbf{0}$). See figure 2.

Why is F a projective function? Let U and V be the linear subspaces representing p and π, respectively, in the homogeneous model. Note that U and V are complementary subspaces of $\mathbf{R}^4$. Now consider the linear map M that decomposes each vector x of $\mathbf{R}^4$ into its U and V components, and returns the latter. That is, M projects every vector x into V in a direction parallel to U. It is easy to check that F is precisely the projective function induced by M.

A particularly interesting case is when p and π are polar complementary

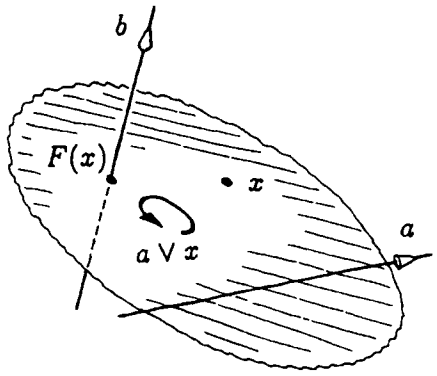

Figure 2. Projecting $\mathbf{T}_3$ onto a line.

flats of $\mathbf{T}_\nu$, with $p = \pi^\dashv$. In that case I call F the *polar projection* on π. In the homogeneous model, F corresponds to orthogonal projection of $\mathbf{R}^n$ onto the linear subspace representing the flat π.

1.3. Properties of projective functions

It follows immediately from the definition that projective functions are closed under composition. Also, if F is a projective function with domain S, then the restriction of F to any subflat X of S is also projective.

Like proper maps, projective functions satisfy the property $(a \vee b)M = (aM) \vee (bM)$ except that M is allowed to produce the zero flat even when the argument is nonzero. The analogous formula for $\wedge$ is not valid, at least not when stated so directly, essentially because the universe of the domain flat S is not mapped to that of the range flat T. However, if X is any flat of S disjoint from the null space of M, then the image $Y = XM$ is a well-defined flat of T, and the restriction of M to X is a non-degenerate projective map from X to XM. In that case, for two flats a, b of X we have

$$(a \wedge_X b)M = (aM) \wedge_Y (bM).$$

In particular, we can take X of maximum rank subject to $X \cap \mathit{Null}(M) = \emptyset$, that is, we can let X be any flat of S that is complementary to $\mathit{Null}(M)$. In that case Y will cover the actual range of M, that is, $SM = Y$ (as sets of points).

1.4. Natural domain of projective functions

Given a projective function defined on $\mathbf{T}_\nu$ (or any flat thereof) we can use the standard polarity relation to pick a particular flat set X of maximum rank such that the restriction of M to X is non-degenerate. That set is simply the polar complement of the function's null space, $(\mathit{Null}(M))^\perp$. I call that set the *natural domain* of M, and denote it by $\mathit{Dom}(M)$. The restriction of M to $\mathit{Dom}(M)$ is non-degenerate and

has range $Range(M)$.

Conversely, any non-degenerate map M defined on a flat X of $\mathbf{T}_\nu$ can be extended to a projective function N, from the whole $\mathbf{T}_\nu$ onto $Dom(M)$, consisting of a polar projection of $\mathbf{T}_\nu$ onto X followed by the given map M. The function N is the *polar extension* of M to $\mathbf{T}_\nu$. In particular, the polar projection of $\mathbf{T}_\nu$ onto a flat X is the polar extension of the identity map on X.

1.5. The inverse of a projective function

Since projective functions may be many-to-one, they do not have an inverse in the ordinary sense. However, if M is a projective function from $\mathbf{T}_\nu$ into $\mathbf{T}_\mu$, we can derive a non-degenerate map N from it as described above, and extend the inverse of N to a projective function from $\mathbf{T}_\mu$ into $\mathbf{T}_\nu$. In analytic terms, the inverse defined this way corresponds to the so-called *generalized least-squares inverse* or *Moore-Penrose inverse* of the coefficient matrix M of M. This is the unique matrix $\overleftarrow{M}$ such that

$$(M\overleftarrow{M})^{\mathrm{tr}} = M\overleftarrow{M} \qquad M\overleftarrow{M}M = M$$
$$(\overleftarrow{M}M)^{\mathrm{tr}} = \overleftarrow{M}M \qquad \overleftarrow{M}M\overleftarrow{M} = \overleftarrow{M}.$$

These conditions essentially say that $M\overleftarrow{M}$ and $\overleftarrow{M}M$ are orthogonal projections onto the column space and the row space of M, respectively.

It is worth noting that the generalized inverse above does not satisfy the equation $xM\overleftarrow{M} = x$, unless $x \in Dom(M)$. For the same reason, we don't have $\overleftarrow{MN} = \overleftarrow{N}\,\overleftarrow{M}$, unless $Range(M) = Dom(N)$.

1.6. Topological properties

Projective functions that are not one-to-one obviously are not homeomorphisms, but they still preserve some of the topological structure of the projected objects. For one thing,

Theorem 1. *A projective function F from a space D into a space R maps closed subsets of $D \setminus Null(F)$ to closed subsets of R.*

PROOF: In terms of the spherical model of $\mathbf{T}_\nu$, a projective function F is a linear function F followed by central projection onto the unit sphere. As long as the argument is not in the null space of F, these operations are well-defined and continuous. Therefore, F is continuous on the set $D^+ = D \setminus Null(F)$, which means F maps closed subsets of D^+ to closed subsets of its range $F(D^+)$. But $F(D^+)$ is a flat set of R, which is closed in R. Therefore, sets that are closed relative to $F(D^+)$ are also closed in R.

QED.

The following theorem is a bit less trivial:

Theorem 2. *A projective function F of a space D onto a space R maps open subsets of $D \setminus Null(F)$ to open subsets of R.*

PROOF: Let X be an open subset of $D \setminus Null(F)$. If X is empty, $F(X)$ is empty and the theorem is vacuously true. If X is not empty, then $Null(F)$ is a proper subset of D. Let then x be any point of X, and let Y be a subspace of D with maximum rank such that $x \in Y$ and $Y \cap Null(F) = \emptyset$. It is easy to see that Y must be a complementary flat for $Null(F)$, which means F maps Y onto R in a one-to-one fashion. Since F is continuous outside $Null(F)$, it is a homeomorphism of Y onto R, and maps open sets of the former to open sets of the latter. Since X is open, $X \cap Y$ is open relative to Y, so $F(X \cap Y)$ is open relative to R. Since $x \in X \cap Y \subseteq X$, we also have $F(x) \in F(X \cap Y) \subseteq F(X)$.

This shows that every point $F(x)$ of $F(X)$ is contained in some subset of $F(X)$ that is open in R. This means $F(X)$ is an open subset of R.

QED.

2. Computer representation

In chapter 8 we saw that a projective map from $\mathbf{T}_\nu$ to $\mathbf{T}_\nu$ or $\neg\mathbf{T}_\nu$ can be represented by an $n \times n$ matrix of coefficients

$$\left[\!\left[\begin{array}{ccc} m_0^0 & \cdots & m_\nu^0 \\ \vdots & & \vdots \\ m_0^\nu & \cdots & m_\nu^\nu \end{array} \right]\!\right] \tag{2}$$

with nonzero determinant. Projective functions have the same representation, except that the matrix may have zero determinant. In fact, we have

Theorem 3. *Every projective function of $\mathbf{T}_\nu$ into itself is completely described by an $n \times n$ real matrix. The function is degenerate if and only if the matrix has zero determinant.*

As in the case of projective maps, two matrices determine the same map if and only if they are positive multiples of each other.

2.1. Matrix of a perspective projection

An important example is the perspective projection M defined by $xM = (a \vee x) \wedge b$, where a and b are complementary flats of $\mathbf{T}_\nu$. If we know positive simplices $a^0; .. a^\kappa$ for a and $b^0; .. b^\mu$ for b, we can compute the matrix of M by the formula $M = \overleftarrow{C}B$, where B and C are the $n \times n$ matrices

$$B = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ b_0^0 & \cdots & \cdots & b_\nu^0 \\ \vdots & & & \vdots \\ b_0^\mu & \cdots & \cdots & b_\nu^\mu \end{pmatrix} \qquad C = \begin{pmatrix} a_0^0 & \cdots & \cdots & a_\nu^0 \\ \vdots & & & \vdots \\ a_0^\kappa & \cdots & \cdots & a_\nu^\kappa \\ b_0^0 & \cdots & \cdots & b_\nu^0 \\ \vdots & & & \vdots \\ b_0^\mu & \cdots & \cdots & b_\nu^\mu \end{pmatrix} \tag{3}$$

Briefly, the effect of $\overleftarrow{C}$ is to compute the coordinates of a given vector u of $\mathbf{R}^n$ relative to the basis formed by the vectors $a^0; .. a^\kappa; b^0; .. b^\mu$. The subsequent multiplication by B throws away the a^i components of u, and collects the b^i components. The correctness of the formula $M = [\![\overleftarrow{C}B]\!]$ follows readily from this observation.

2.2. Domain, range, and null space

Let M be a projective function with coefficient matrix M, and let $\mathbf{e}^0 = (1, 0, 0, .. 0)$, $\mathbf{e}^1 = (0, 1, 0, .. 0)$, ..., $\mathbf{e}^\nu = (0, 0, 0, .. 1)$ be the canonical basis of $\mathbf{R}^n$. Observe that row i of matrix M homogeneous coordinates of the image of point $[\mathbf{e}^i]$ through M. In general, the coordinates of the image of any point x of $\mathbf{T}_\mu$ by M will be a linear combination of the rows of M. Therefore,

Theorem 4. *The range of a projective function from $\mathbf{T}_\mu$ into $\mathbf{T}_\nu$ is the the flat set of $\mathbf{T}_\nu$ corresponding to the linear subspace spanned by the rows of its coefficient matrix.*

Observe also that the jth coordinate of xM is the dot product of the coordinate vector of x by the jth column of matrix M. Therefore, a point of $\mathbf{T}_\mu$ is mapped to $\mathbf{0}$ if and only if its coordinate vector is orthogonal to every column of M. In other words, the null space of the linear map with matrix M is the orthogonal complement in $\mathbf{R}^m$ of the subspace spanned by the columns of M. It follows that

Theorem 5. *The natural domain of a projective function from $\mathbf{T}_\mu$ into $\mathbf{T}_\nu$ is the flat set of $\mathbf{T}_\mu$ corresponding to the subspace spanned by the columns of its coefficient matrix.*

As we know from linear algebra, the row and column spaces of a matrix have the same dimension, which is the size of the largest submatrix with non-zero determinant. This number is the *rank* of the map, and coincides with the geometric rank of the flat sets *Range*(M) and *Dom*(M). When the rank is equal to the number of rows m, the map is one-to-one; when the rank is equal to the number of columns n, the map ranges over the whole $\mathbf{T}_\nu$.

In light of the above, an $m \times n$ matrix M can be taken to represent either a (possibly degenerate) projective function M from $\mathbf{T}_\mu$ into $\mathbf{T}_\nu$, or a (non-degenerate) projective map N from *Dom*(M) to *Range*(M). Fortunately, in most practical situations (for example, when implementing a basic geometric operations library) it is not necessary to worry about this distinction, since mapping a point x through M or N is done by the same formulas and using the same coefficient matrix. Therefore, both operations can be implemented by a single procedure.

In the same vein, we can use a single procedure to compute both the polar inverse $\overleftarrow{M}$ of M (which is a projective function from $\mathbf{T}_\nu$ into $\mathbf{T}_\mu$) and the plain inverse $\overleftarrow{N}$ of N (a proper map from *Range*(M) to *Dom*(M)). Algorithms for computing the matrix of $\overleftarrow{M}$ can be found in the numerical analysis literature.[1,2]

2.3. The canonical embedding map

It is often useful to identify the space $\mathbf{T}_\mu$ (for all $\mu < \nu$) with the flat S of $\mathbf{T}_\nu$ generated by the first m points of the standard simplex of $\mathbf{T}_\nu$. This flat consists of all points of $\mathbf{T}_\nu$ whose last $n - m$ coordinates are zero. The *canonical embedding* of $\mathbf{T}_\mu$ into $\mathbf{T}_\nu$ is the function η that takes every point of $\mathbf{T}_\mu$ to the "same" point of S. In analytic terms, η simply appends $n - m$ zeros to the homogeneous coordinates of its argument; its matrix is

$$\begin{matrix} \overbrace{}^{m} & \overbrace{}^{n-m} \end{matrix}$$
$$\begin{bmatrix} 1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1 & 0 & \cdots & 0 \end{bmatrix} \tag{4}$$

The inverse map $\overleftarrow{\eta}$ is the polar projection of $\mathbf{T}_\nu$ onto S, which simply throws away the last $n - m$ coordinates of its argument. The matrix of $\overleftarrow{\eta}$ is the transpose of (4).

2.4. Alternative representations

The coefficient matrix is the most natural representation of a projective map, but not necessarily the most convenient. For one thing, computing the image of a point x under the inverse map $\overleftarrow{M}$ is far more expensive than computing the direct image. To do that we essentially have to solve a system of n linear equations on n unknowns, which requires $O(n^3)$ operations (instead of the $O(n^2)$ required by the direct mapping). Computing the image of a hyperplane h given its coefficients is equally expensive.

One way to reduce this cost is to precompute the inverse matrix $\overleftarrow{M}$ (or the adjoint), and carry it along with the matrix M. Then points and hyperplanes can be mapped equally fast, in $O(n^2)$ time. One disadvantage of this idea is that it takes twice as much space: in the case of $\mathbf{T}_3$, it requires 32 real numbers instead of 16. Besides consuming more memory space, this enlarged representation also takes longer to build, copy, and compose.

2.5. The LU factorization

We can get the best (and worst?) of both worlds by storing the matrix M in some compact factored form such that mapping by both M and $\overleftarrow{M}$ can be performed relatively fast. For example, we know from linear algebra that any $m \times n$ matrix with $m \leq n$ can be factored into the product of a row permutation matrix, an $m \times m$ lower triangular matrix, and an $m \times n$ upper triangular matrix:

$$M = LU = \begin{pmatrix} l_0^0 & & 0 \\ \vdots & \ddots & \\ l_0^\mu & \cdots & l_\mu^\mu \end{pmatrix} \begin{pmatrix} \overbrace{u_0^0 \cdots u_\mu^0}^{m} & \overbrace{\cdots \cdots u_\nu^0}^{n-m} \\ \ddots & & \vdots \\ 0 & u_m^\mu u \cdots \cdots & u_\nu^\mu \end{pmatrix}$$

This *Gaussian LU factorization* can be computed in $O(mn^2)$ time, can be represented in $mn + O(m+n)$ words of storage, and still allows the mapping of points to be performed in $O(mn)$ time. If the matrix is square ($m = n$), the inverse mapping too can be computed from the factored form at roughly the same cost as the direct one.

2.6. The singular value decomposition

An alternative to the Gaussian LU decomposition that is worth considering is the *singular value decomposition* (SVD). An arbitrary $m \times n$ real matrix M (with $m \leq n$) can always be factored as the product of three matrices $U \in \mathbf{R}^{m \times m}$,

$\Sigma \in \mathbf{R}^{m \times n}$, and $V \in \mathbf{R}^{n \times n}$ such that U and V are orthogonal, and Σ is all zero except for the elements on the main diagonal:

$$M = U\Sigma V^{\mathrm{tr}} = \begin{pmatrix} u_0^0 & \cdots & u_\mu^0 \\ \vdots & & \vdots \\ u_0^\mu & \cdots & u_\mu^\mu \end{pmatrix} \begin{pmatrix} \overbrace{\begin{matrix} \sigma_0 & & 0 \\ & \ddots & \\ 0 & & \sigma_\mu \end{matrix}}^{m} & \overbrace{\begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix}}^{n-m} \end{pmatrix} \begin{pmatrix} v_0^0 & \cdots & \cdots & v_0^\nu \\ \vdots & & & \vdots \\ & & & \\ & & & \\ \vdots & & & \vdots \\ v_\nu^0 & \cdots & \cdots & v_\nu^\nu \end{pmatrix}$$

The case $m > n$ is similar, except that the matrix Σ will have more rows than columns. It is always possible to arrange for U and V to have positive determinant, and for the numbers $|\sigma_i|$ (the *singular values* of the matrix M) to be sorted in non-increasing order. Furthermore, we can arrange for all σ_i to be non-negative, except perhaps for σ_0. Algorithms for computing this decomposition in time roughly $O(mn(m+n))$ are well documented in the numerical analysis literature.[1,2]

Compared to the LU decomposition, the SVD has the advantage of treating domain and range in a more symmetric fashion. In fact the generalized inverse of M is $V\overleftarrow{\Sigma}U^{\mathrm{tr}}$, where $\overleftarrow{\Sigma}$ is the transpose of Σ with every nonzero σ_i replaced by $1/\sigma_i$. Therefore, the SVD allows points to be mapped through both M *and its inverse* at the same cost, even when the matrix M is non-square or singular.

Note that when $m \geq n$ we only have to store the first m rows of V. More precisely, if the map's range is a flat set of rank r, then only the first r of the σ_i will be non-zero, which means we only have to store the first r columns of U and the first r rows of V. Therefore the SVD can be represented in $r(n+m)+O(m+n)$ floating-point words, while still allowing points to be mapped in $O(r(m+n))$ time. In fact, by representing the matrices U and V as the product of Householder reflections[1:3.3] it is possible to bring the storage cost down to $r(m+n)-r^2+O(m+n)$, without increasing the asymptotic cost of point mapping by more than a constant factor. These optimizations make the SVD a reasonable alternative to LU decomposition for general maps, and a definite win for highly degenerate maps.

Most of the properties of the map M that are related to the generalized inverse are readily obtainable from the singular value decomposition of its matrix. For example, the rank of M is the number of non-zero σ_i. The null space of M is the space spanned by the rows of U whose corresponding σ_i are zero. The SVD decomposition has also good numerical properties when computations are carried out in floating point. The main disadvantage of the SVD is that it takes somewhat longer to compute, and it makes map composition substantially harder.

One can imagine many other possible representations for projective maps,

based on other factorizations or more exotic schemas. Each representation has its advantages and disadvantages, and its merit depends, among other things, on the relative frequency of the various operations for which it is used. At this point, I can only say that determining the "best" representation of projective maps for general use is still an open problem.

3. References

[1] G. H. Golub and C. F. Van Loan: **Matrix computations.** Johns Hopkins University Press, Baltimore (1983).

[2] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling: **Numerical recipes: The art of scientific computing.** Cambridge University Press, Cambridge (1986).

Chapter 12
Projective frames

Frames play the same role in projective geometry as bases do in linear algebra and as coordinate systems do in physics and Cartesian geometry. Informally, a frame is a geometric object that can be used as a reference in order to assign unique and unambiguous numeric coordinates to every point of some space.

An even more important use of frames is in the description of projective maps for input to programs and subroutines. Instead of writing down the transformation matrix, it is generally much easier for the user or programmer to give a pair of frames, and ask for the map that takes one frame to the other. As we shall see, such a map exists (and is unique) if and only if the corresponding parts of the two frames have the same relative orientation.

1. Nature of projective frames

Recall that in the homogeneous model a projective map between two subspaces R, S of $\mathbf{T}_\nu$ with rank k is a linear map between two k-dimensional linear spaces U, V of $\mathbf{R}^n$. As we know from linear algebra, such a map is completely specified by giving k independent vectors in U and their images in V. From this observation we may be led to think that a projective map from R to S can be completely specified by giving k independent points (i.e., a proper simplex) on R and their images on S. Unfortunately, this is not the case. The problem is that a point of S specifies the direction of a vector of V, but not its length. For example, suppose we want a map of $\mathbf{T}_1$ to itself that maps

$$\begin{array}{ll} a = [1,0] & \\ b = [0,1] & \end{array} \quad \text{to} \quad \begin{array}{l} p = [1,1] \\ q = [3,5] \end{array}$$

An obvious choice is the map

$$M = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}$$

However, the point p can be written also as $[2,2]$, so the map

$$N = \begin{bmatrix[2 & 2 \\ 3 & 5 \end{bmatrix]$$

also takes $(a;b)$ to $(p;q)$. Yet the two maps are different, since, for example, $[2,1]M = [5,7]$, whereas $[2,1]N = [7,9]$.

Obviously, there are infinitely many projective maps of $\mathbf{T}_1$ to itself that take $(a;b)$ to $(p;q)$. To make the map unique, we must give some additional information. As we shall see, one extra point and its image are enough to completely specify the projective map. In other words, a frame for a κ-dimensional projective space must have at least $(\kappa + 2)$ points.

Instead of an additional point, we can specify a hyperplane of R and its desired image in S. Such "mixed" frames may seem less natural than the all-point ones, but in fact they have many computational and geometric advantages. For one thing, such frames arise quite naturally when dealing with affine maps.

The moral of the story is that there is no obvious choice as to what should be a projective frame. Rather, there are several types of frames, and each type has its uses and advantages. This situation is quite different from the one in linear algebra or Cartesian geometry, where the concept of "frame" has a unique natural definition. In projective geometry that concept must be defined in a more general and abstract way, as done below.

1.1. Arrangements

An *arrangement* is any finite ordered sequence $a = (a^0, a^1, \ldots, a^\mu)$ of flats in a space S. The *span* of an arrangement a is the flat set $Span(a)$ of S with smallest dimension that contains every element of a. The *dimension* and the *rank of* a are those of its span.

A κ-dimensional simplex is an example of an arrangement of rank k. Another example of arrangement is a list of two points p, q and one line l; its rank can be 2, 3, or 4, depending on the relative positions of p, q, and l.

1.2. Similar and categorical arrangements

We are interested in arrangements that can be used to specify projective maps between two spaces. We would like to be able to unambiguously specify a projective map from a space R to a space S by giving some arrangement a on R and its desired image b on S.

Of course, the arrangements cannot be arbitrary. First of all, they must be *projectively similar*: that is, there must exist some projective map M such that

$M(a^i) = b^i$ for all i. For example, two proper simplices with the same number of vertices are similar. This relation is obviously symmetric, reflexive, and transitive.

Secondly, the map must be unique. Let's say that an arrangement a is *categorical* if every projective map that takes a to itself also takes every point of $Span(a)$ to itself. This property implies the one we are after, namely

Theorem 1. *If a and b are similar categorical arrangements, there is exactly one projective map from $Span(a)$ to $Span(b)$ that takes a to b.*

PROOF: Since a and b are similar, there is some projective map that takes a to b. Its domain is a flat set that contains a, and therefore $Span(a)$. Let M be the restriction of that map to $Span(a)$. The range of M is a flat set containing b, and therefore it contains $Span(b)$. Moreover, $\overleftarrow{M}(Span(b))$ is a flat set containing a; it follows that the range of M is exactly $Span(b)$.

Now let G be any projective map from $Span(a)$ to $Span(b)$ that takes a to b. The composition $M\overleftarrow{G}$ takes a to itself, and therefore must take every point of $Span(a)$ to itself. It follows $\overleftarrow{G}$ is the inverse of M, i.e. $M = G$. We conclude the map M is unique.

QED.

2. Frames

Definition 1. A *frame* for a flat set S is a categorical arrangement whose span is exactly the set S.

Theorem 1 justifies this definition, which has an obvious corollary:

Theorem 2. *The image of a frame by a projective map is a similar frame.*

Recall that any two oriented projective spaces with same dimension are related by some projective map. Therefore, if a is a frame for $\mathbf{T}_\kappa$ with $\kappa \leq \nu$, then every κ-dimensional subspace of $\mathbf{T}_\nu$ has a frame similar to a.

2.1. Classification of frames

The definition of frame given above is quite general, to such an extent that it does not fix the number and rank of the frame elements. In particular, if $a = (a^0, .. a^\mu)$ is a frame, and f is any flat obtained from $a^0, .. a^\mu$ by join and relative complement operations, then the arrangement $(a^0, .. a^\mu, f)$ is also a frame, with same span as a. Obviously, there are many "flavors" (similarity classes) of projective

frames, even for a fixed flat set S. What is more, there is no single "flavor" that is convenient for all applications.

The general problem of characterizing similar and categorical arrangements is relatively well-studied but is rather hard. The paper by Crapo and Ryan[1] is an example of recent work in this area. Fortunately, for the purpose of implementing a library of basic geometric operations it is enough to consider a couple of the most important classes of frames.

2.2. Frame type

In order for two frames (or in general two arrangements) to be similar, they must first of all have the same *type*, that is, must have the same number of elements, and corresponding elements must have the same rank.

This condition is necessary, but obviously not sufficient. For example, a degenerate triangle has the same type as a proper one, but the two are not similar. Nevertheless, the type of a frame is an obvious attribute to use in their classification. The frames I will consider in detail are of two types: *point frames* and *mixed frames*.

2.3. Point frames

A *point frame* for a flat set S of rank k is an arrangement of $k+1$ points such that any k of them form a proper simplex of S. For reasons that will become clear later on, I will call the first k points the *main simplex*, and the last one the *unit point*. For example, point frames for $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$ consist of, respectively, three points (pairwise unrelated), four points (no three of them collinear), and five points (no four of them on the same plane). See figure 1.

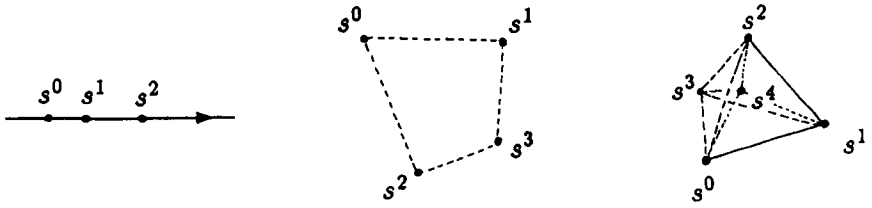

Figure 1. Point frames for $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$.

2.4. Mixed frames

A *mixed frame* for a flat set S is an arrangement $(s^0, .. s^\kappa, h)$ where $(s^0; .. s^\kappa)$ is a non-degenerate simplex of S, and h is an oriented hyperplane of S that avoids all vertices of that simplex. The hyperplane h is the *horizon* of the frame, and $(s^0; .. s^\kappa)$ is the *main simplex*. For example, a mixed frame for $\mathbf{T}_2$ consists of three

points p, q, r forming a proper triangle, and a line l that does not pass through any of those points. See figure 2.

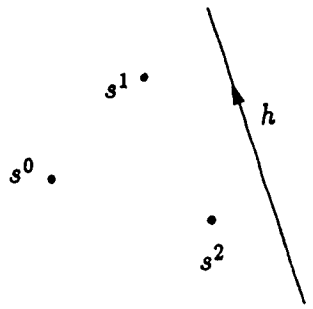

Figure 2. A mixed frame for $\mathbf{T}_2$.

In what follows we will justify these definitions, by showing that point frames and mixed frames are indeed categorical arrangements. To do that, we must examine first the conditions for two such frames to be similar.

2.5. Orientented span

Although we generally define the span of an arrangement as an *unoriented* flat set, in the case of point frames and mixed frames we can give that set an unambiguous orientation, as determined by the frame's main simplex. More precisely, if f is a point frame or mixed frame with main simplex $(s^0; .. s^\kappa)$, then the *oriented span* of f is the flat $s^0 \vee s^1 \vee \cdots \vee s^\kappa$.

Therefore, for any *oriented* flat S, we can distinguish the *positive* point frames (whose oriented span is S) from the *negative* ones (whose oriented span is $\neg S$). Ditto for mixed frames. Note that this is not an intrinsic property of the frame: a positive frame for a space S is also a negative frame for the space $\neg S$.

2.6. Signature of a point frame

Let $f = (s^0, .. s^\kappa, u)$ be a point frame. For each i in turn, consider the simplex obtained by replacing vertex s^i of the main simplex by the unit point u. Let σ_i be the orientation (+ or −) of this simplex relative to the main one. The sequence $\sigma^0 .. \sigma^\kappa$ is by definition the *signature* of the frame f.

In other words, the signature of a point frame is the signature of its unit point relative to its main simplex. Element σ^i of the signature tells whether u and x^i are on the same side of the hyperplane of S determined by the remaining points. In particular, the signature is $++\cdots+$ if the unit point is inside the main simplex,

and $--\cdots-$ if it is inside the antipodal simplex. See figure 3.

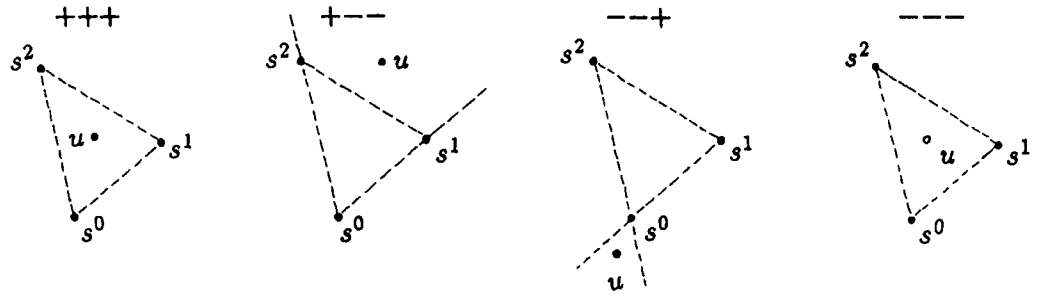

Figure 3. Point frames of $\mathbf{T}_2$ with various signatures.

2.7. Signature of a mixed frame

Let $f = (s^0, .. s^\kappa, h)$ be a mixed frame. The *signature* of f is the sequence $\sigma_0 .. \sigma_\kappa$, where $\sigma_i \in \{\pm 1\}$ is the relative position of the flats s^i and h in the oriented span of f. In other words, σ_i is such that

$$s^i \vee h = \sigma_i \circ (s^0 \vee s^1 \vee \cdots \vee s^\kappa) \quad \text{for all } i.$$

Notice how the signature of a frame f is defined solely in terms of the orientations of the elements of f and their joins, with no reference to any independently supplied orientation for the spanned flat set. It follows that signatures are preserved by arbitrary projective maps. That includes maps from the oriented span of f to its opposite. A positive frame for a space S is also a negative frame for the space $\neg S$, but its signature is the same in both cases. In fact, for any signature σ and any space S, there are both positive and negative frames for S with signature σ. See figure 4.

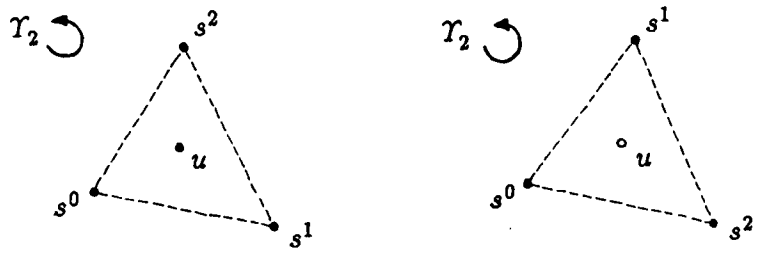

Figure 4. Positive and negative frames for $\mathbf{T}_2$ with signature +++.

3. Standard frames

The manipulation of projective maps and frames is often simplified by the choice of "standard" reference frames for $\mathbf{T}_\nu$. For example, suppose we want to compute the matrix of the projective map that relates two arbitrary point or mixed frames a, b. One way to solve that problem is to compute the maps M_a and M_b that take some standard frame f to a and b. The desired map will then be the product $\overleftarrow{M}_a M_b$. Obviously, for the maps to exist all three frames must have the same rank, type, and signature. Therefore we need at least one "standard" frame for each combination of these attributes.

3.1. Standard point frames

A standard point frame for $\mathbf{T}_\nu$ must consist of $\nu + 2$ points such that any $\nu + 1$ of them form a proper simplex. An obvious choice is to take the main simplex $(\mathbf{e}^0; .. \mathbf{e}^\nu)$ and the *standard unit point* $\mathbf{u} = [1, 1, 1, .. 1]$. See figures 5 and 6.

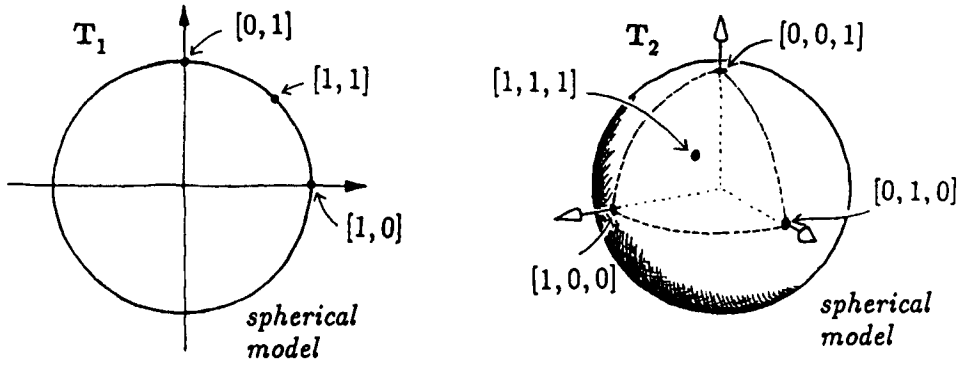

Figure 5. The standard point frames of $\mathbf{T}_1$ and $\mathbf{T}_2$ with signature $+\cdots+$.

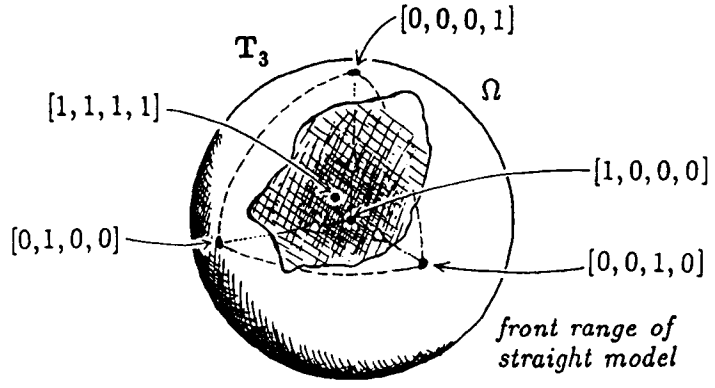

Figure 6. The standard point frame of $\mathbf{T}_3$ with signature $++++$.

In the straight model, this frame consists of the front origin O, the points at infinity on each axis, and the point $(1,1,..1)$, all on the front range. This frame has signature $++\cdots+$, meaning its unit point is inside its main simplex. To obtain standard frames with other signatures, it suffices to replace u by other suitably located points. I will define the *standard point frame with signature* σ, denoted by pfr_σ, as consisting of the canonical simplex $\mathbf{e}^0,..\mathbf{e}^\nu$, plus the point $[\sigma] = [\sigma^0,..\sigma^\nu]$ whose homogeneous coordinates are the desired signature. In the straight model, this point has Cartesian coordinates $(\sigma_1/\sigma_0, \sigma_2/\sigma_0, \ldots, \sigma_\nu/\sigma_0)$, and lies on the front or back range depending on whether σ_0 is $+1$ or -1. See figure 7.

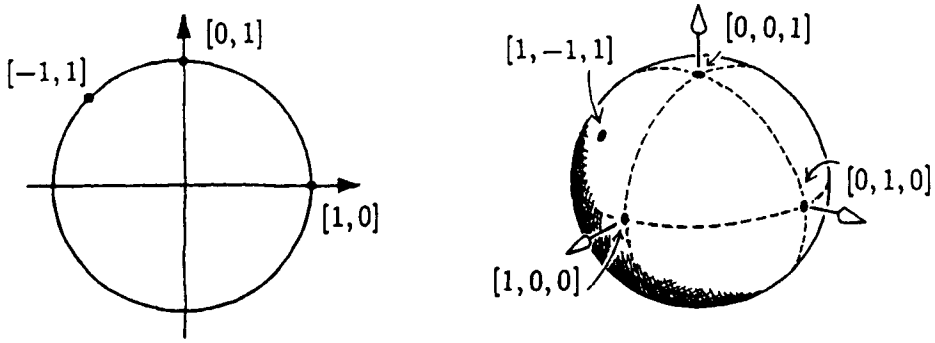

Figure 7. The standard point frames pfr_{-+} of $TT1$ and pfr_{+-+} of $\mathbf{T}_2$.

3.2. Standard mixed frames

Among the mixed frames of $\mathbf{T}_\nu$ with signature $++\cdots+$, the most obvious choice is the frame consisting of the canonical simplex together with the hyperplane $\xi = \langle 1,1,1..1\rangle$. In the straight model, the hyperplane ξ is perpendicular to the vector $(1,1,..1)$ of the front range, passes through the point $(-1/\nu, -1/\nu, ..-1/\nu)$ of the negative orthant, and is oriented clockwise as seen from the front origin. See figures 8 and 9.

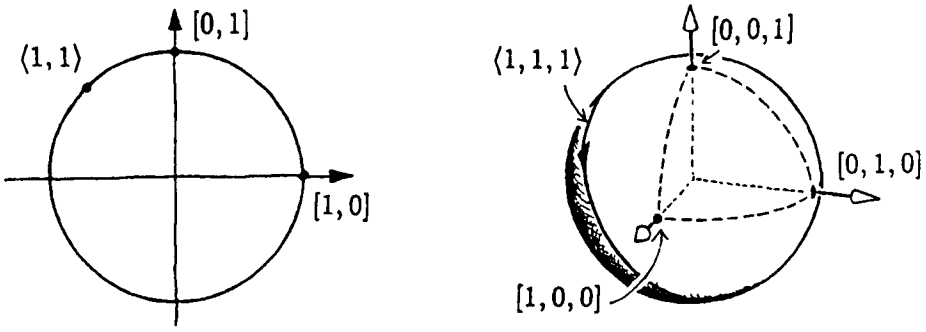

Figure 8. The standard mixed frames of $\mathbf{T}_1$ and $\mathbf{T}_2$ with all-positive signature.

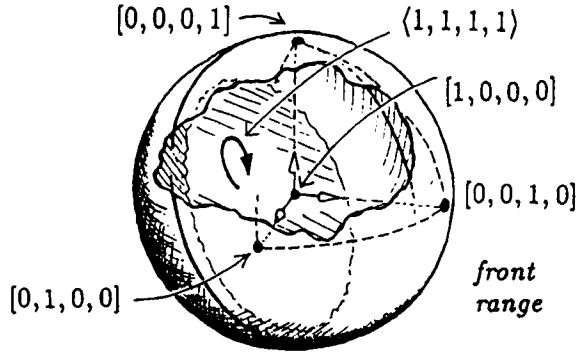

Figure 9. The standard mixed frame of $\mathbf{T}_3$ with all-positive signature.

As in the case of point frames, we can obtain frames of arbitrary signature by changing only the horizon hyperplane. I define the *standard mixed frame with signature* σ as consisting of the canonical simplex $(\mathbf{e}^0; .. \mathbf{e}^\nu)$, and the hyperplane $\langle\sigma\rangle = \langle\sigma_0, .. \sigma_\nu\rangle$ whose coefficients are the desired signature. See figure 10.

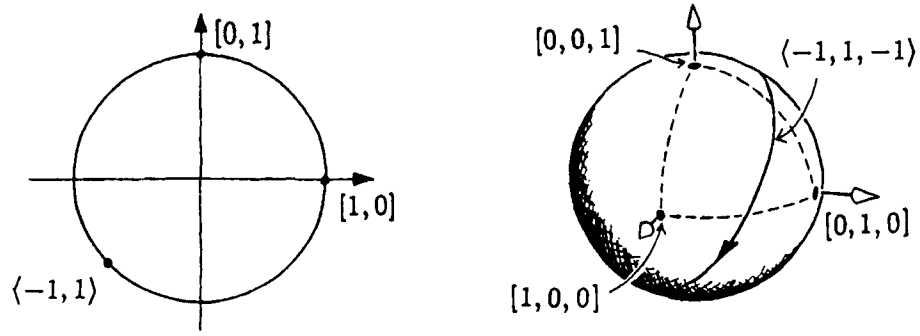

Figure 10. Some standard mixed frames of $\mathbf{T}_1$ and $\mathbf{T}_2$: (a) mfr_{-+}, (b) mfr_{-+-}.

3.3. Mapping to the standard frames

We can now prove that every point frame or mixed frame is similar to a standard one. Let's characterize first the maps that take the canonical simplex $(\mathbf{e}^0; .. \mathbf{e}^\nu)$ to a given simplex $s = (s^0; .. s^\nu)$ of $\mathbf{T}_\nu$ (positive or negative). It is not hard to see that such maps are precisely those of the form

$$\begin{bmatrix} \gamma_0 & & 0 \\ & \gamma_1 & \\ & \ddots & \\ 0 & & \gamma_\nu \end{bmatrix} \begin{bmatrix} s_0^0 & s_1^0 & \cdots & s_\nu^0 \\ s_0^1 & s_1^1 & \cdots & s_\nu^1 \\ \vdots & & & \vdots \\ s_0^\nu & s_1^\nu & \cdots & s_\nu^\nu \end{bmatrix} \tag{1}$$

where the γ_i are positive but otherwise arbitrary. The theorems below show that the scale factors γ_i give enough degrees of freedom for us to specify the image of an extra point or hyperplane. That is, we can adjust the map (1) so that it also takes care of the last item (unit point or horizon) of the two frames.

Theorem 3. *Every point frame of* $\mathbf{T}_\nu$ *is similar to the standard one with the same signature.*

PROOF: Let $f = (s^0, .. s^\nu, u)$ be a point frame with signature $\sigma = \sigma_0, .. \sigma_\nu$. Let $s^i = [s^i_0, .. s^i_\nu]$ and $u = [u^0, .. u^\nu]$, and let $(\alpha_0, .. \alpha_\nu)$ be the solution to the linear system

$$(\alpha_0, .. \alpha_\nu) \begin{pmatrix} s^0_0 & \cdots & s^0_\nu \\ \vdots & & \vdots \\ s^\nu_0 & \cdots & s^\nu_\nu \end{pmatrix} = (u_0, .. u_\nu). \tag{2}$$

Note that since $(s^0; .. s^\nu)$ is a non-degenerate simplex, the matrix above has non-zero determinant. According to Cramer's rule the solution of (2) is

$$\alpha_i = \begin{vmatrix} s^0_0 & \cdots & \cdots & \cdots & s^0_\nu \\ \vdots & & & & \vdots \\ u_0 & \cdots & \cdots & \cdots & u_\nu \\ \vdots & & & & \vdots \\ s^\nu_0 & \cdots & \cdots & \cdots & s^\nu_\nu \end{vmatrix} \Big/ \begin{vmatrix} s^0_0 & \cdots & \cdots & \cdots & s^0_\nu \\ \vdots & & & & \vdots \\ s^i_0 & \cdots & \cdots & \cdots & s^i_\nu \\ \vdots & & & & \vdots \\ s^\nu_0 & \cdots & \cdots & \cdots & s^\nu_\nu \end{vmatrix} \tag{3}$$

Note that the signs of the numerator and denominator in (3) give the orientation of the simplices $(s^0; .. s^{i-1}; u; s^{i+1}; .. s^\nu)$ and $(s^0; .. s^\nu)$. It follows that $|\alpha_i| \neq 0$, and $\text{sign}(\alpha_i) = \sigma_i$.

Now consider the map M_f of the form (1) with $\gamma_i = |\alpha_i|$:

$$M = \begin{bmatrix} |\beta_0| & & 0 \\ & \ddots & \\ 0 & & |\beta_\nu| \end{bmatrix} \begin{bmatrix} s^0_0 & \cdots & s^0_\nu \\ \vdots & & \vdots \\ s^\nu_0 & \cdots & s^\nu_\nu \end{bmatrix} \tag{4}$$

This map obviously takes the main simplex of pfr_σ to that of f. I claim M_f also

takes the unit point $[\sigma]$ of pfr_σ to the unit point u of f. Indeed,

$$[\sigma_0,..\sigma_\nu]M_f = [\sigma_0,..\sigma_\nu]\begin{bmatrix}|\alpha_0| & & 0\\ & \ddots & \\ 0 & & |\alpha_\nu|\end{bmatrix}\begin{bmatrix}s_0^0 & \cdots & s_\nu^0\\ \vdots & & \vdots\\ s_0^\nu & \cdots & s_\nu^\nu\end{bmatrix}$$

$$= [\alpha_0,..\alpha_\nu]\begin{bmatrix}s_0^0 & \cdots & s_\nu^0\\ \vdots & & \vdots\\ s_0^\nu & \cdots & s_\nu^\nu\end{bmatrix} = [u_0,..u_\nu]$$

QED.

Theorem 4. *Every mixed frame of $\mathbf{T}_\nu$ is similar to the standard one with same signature.*

PROOF: Let $f = (s^0,..s^\nu,h)$ be a mixed frame of $\mathbf{T}_\nu$ with signature $\sigma = \sigma_0\cdots\sigma_\nu$. Let $s^i = [s_0^i,..s_\nu^i]$ and $h = \langle h^0,..h^\nu\rangle$. Define

$$\lambda = \mathrm{sign}(s^0,..s^\nu) \qquad \beta_i = \frac{1}{\sum_j s_j^i h^j} \tag{5}$$

Observe that the sign of β_i is the relative orientation of s^i and h in $\mathbf{T}_\nu$, and that the numerator λ is the orientation of the span of f relative to $\mathbf{T}_\nu$. Therefore $\mathrm{sign}(\beta_i) = \lambda\sigma_i$.

Now consider the map M_f of the form (1) with $\gamma_i = |\beta_i|$:

$$M_f = \begin{bmatrix}|\beta_0| & & 0\\ & \ddots & \\ 0 & & |\beta_\nu|\end{bmatrix}\begin{bmatrix}s_0^0 & \cdots & s_\nu^0\\ \vdots & & \vdots\\ s_0^\nu & \cdots & s_\nu^\nu\end{bmatrix} \tag{6}$$

I claim M_f takes the standard mixed frame mfr_σ to f. Obviously, M_f takes the main simplex of the former to that of the latter. As for the horizon, in order to prove that $\xi M_f = h$ it is sufficient to show that, for every point $x \in \mathbf{T}_\nu$, we have $x\diamond\langle\sigma\rangle M_f = x\diamond h$. Since M_f is one-to-one, we can replace x in this formula by by xM_f. Since M_f is a map from $\mathbf{T}_\nu$ to $\lambda\circ\mathbf{T}_\nu$, we have $(xM_f)\diamond(\langle\sigma\rangle M_f) = \lambda(x\diamond\langle\sigma\rangle)$. Therefore, all we have to show is that

$$\lambda(x\diamond\langle\sigma\rangle) = (xM_f)\diamond h \quad \text{for all } x \in \mathbf{T}_\nu. \tag{7}$$

Now, on the one hand

$$\lambda(x \diamond \langle\sigma\rangle) = \lambda(\operatorname{sign} \textstyle\sum_i x_i \sigma_i) \tag{8}$$

and, on the other hand,

$$\begin{aligned}(xM_f) \diamond h &= \operatorname{sign} \textstyle\sum_j (xM_f)_j h^j = \\ &= \operatorname{sign} \textstyle\sum_j (\sum_i x_i \, |\beta_i| \, s_j^i) h^j \\ &= \operatorname{sign} \textstyle\sum_i x_i \, |\beta_i| \, (\sum_j s_j^i h^j) \\ &= \operatorname{sign} \textstyle\sum_i x_i \, |\beta_i| \, \frac{\lambda}{\beta_i} \\ &= \operatorname{sign} \textstyle\sum_i x_i \lambda \operatorname{sign}(\beta_i) \\ &= \lambda(\operatorname{sign} \textstyle\sum_i x_i \sigma_i)\end{aligned} \tag{9}$$

By comparing (8) and (9), we conclude that indeed $\langle\sigma\rangle M_f = h$.
QED.

Recall that any κ-dimensional two-sided subspace of $\mathbf{T}_\nu$ can be projectively mapped to $\mathbf{T}_\kappa$, and that projective maps are closed under inversion and composition. Thanks to these results, we can extend theorems 3 and 4 to arbitrary frames and spaces:

Theorem 5. *Two point frames or two mixed frames are similar if and only if they have the same rank and same signature.*

Note that the map M_f defined in theorems 3 and 4 is insensitive to the orientation of the unit point u or the horizon h.

3.4. Computational considerations

The proofs of theorems 3 and 4 give practical methods for computing the map M that takes a standard frame to a given frame f. The matrix of M has the coordinates of the main simplex of f, with each row scaled by an appropriate factor. In the case of a point frame, the factors α_i are found by solving the linear system (2). (As a byproduct, the signs of the α_i give the signature of the frame f). System (2) can be solved in $O(n^3)$ operations, by factoring the matrix (s_j^i) into the Gaussian LU product (or any similar factorization). Scaling each row of L by the corresponding factor $|\alpha_i|$ gives the desired map M, already in factored form.

For a mixed frame, we compute the scale factors α_i by equation (5), and the matrix of M by (1), in $O(n^2)$ operations. Note that we do not need to know the orientation λ of the frame f. From this it would seem that point frames are substantially more expensive to handle than mixed frames. However, as discussed in a previous chapter, whenever one needs a projective map one usually needs also

its inverse (or a factorization that gives both at roughly the same cost). Since this is an $O(n^3)$ process, the difference in cost between the two types of frames is all but erased.

3.5. A note on one-dimensional frames

Recall that points and hyperplanes are the same thing in spaces of dimension 1. It follows that in those spaces point frames are indistinguishable from mixed frames. This gives rise to some ambiguity, since the signature of a one-dimensional frame f (three collinear points) depends on whether we look at it as a point frame or as a mixed frame. For example, the frame pfr_{-+} (figure 7(a)) has signature $-+$ if viewed as a point frame, and $++$ if viewed as a mixed frame. However, this ambiguity is of no consequence. For any such frame f (i.e., for any sequence of three points) there is only one frame of $\mathbf{T}_1$ whose main simplex is $(\mathbf{e}^0, \mathbf{e}^1)$ and is similar to f. This is both a standard point frame and a standard mixed frame of $\mathbf{T}_1$, and in either interpretation it has the same signature as f. As shown below in section 4.3, there is a unique projective map that takes the three points of one frame to those of the other. In conclusion, three points on a line determine the same map from that line to $\mathbf{T}_1$, whether we use the formulas of theorem 3, or those of theorem 4.

4. Coordinates relative to a frame

In Euclidean geometry, coordinate frames are used not so much to define maps as to assign numerical coordinates to every point. Projective frames too have that function: as we show below, a projective frame for a space S assigns to each point of S a homogeneous coordinate tuple.

By definition, the *coordinates of a point p relative to a frame f* are the homogeneous coordinates of the point pN, where $N = \overleftarrow{M}_f$ is the projective map that takes f to the appropriate standard frame g of $\mathbf{T}_\nu$. The *coefficients of a hyperplane w relative to f* are defined the same way, by mapping w through M and taking the coefficients of the result.

Obviously, the coordinates of the vertices of f's main simplex, relative to f, will be $[1,0,0,..,0]$, $[0,1,0,..,0]$, $[0,0,1,..,0]$, and so on. If f is a point frame, its unit point u will have coordinates $[\sigma_0,..\sigma_\nu]$ relative to f. If f is a mixed frame, its

horizon will have relative coordinates $\langle \sigma_0, .. \sigma_\nu \rangle$ relative to f. See figure 11.

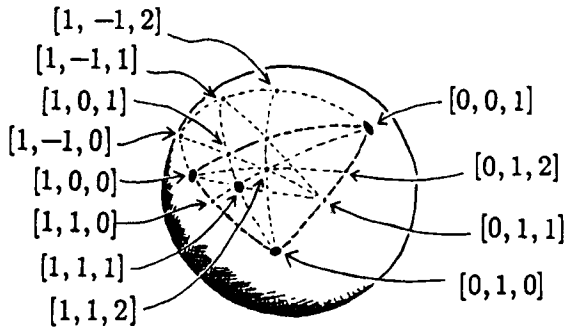

Figure 11. Coordinates relative to a given point frame.

Note that the coordinates relative to a point frame f do not depend on the orientation of the frame's unit point u. That is, if g has the same main simplex as f but unit point $\neg u$, the coordinates relative to f and relative to g are the same. Indeed, we can replace u by any of 2^n other points without affecting the map M_f or the coordinates relative to f. Those are the points whose coordinates relative to f are all ± 1.

4.1. Invariance of relative coordinates

Relative coordinates are invariant under projective maps, in the following sense. Let f be a frame, and M the projective map that takes f to a standard frame g of $\mathbf{T}_\nu$. By definition, the coordinates of a point p relative to f are those of pM. For any map N, the map that takes frame fN to g is $\overleftarrow{N}M$. Therefore the coordinates of pN relative to the frame fN (with respect to the same standard g) are those of $(pN)\overleftarrow{N}M = pM$. That is, for any map N, the coordinates of pN relative to fN are those of p relative to f.

4.2. The center-of-mass interpretation

The coordinates relative to a point frame f have a relatively simple interpretation in terms of the straight model. Suppose that the main simplex s of f lies on the front range of the straight model. Suppose also that the frame f has signature $++\cdots+$, that is, its unit point u lies inside the simplex s. Now imagine that we place a set of "weights" $\gamma_0, .. \gamma_\nu$ at the vertices of the main simplex, in such a way that their center of mass falls on the unit point u. Then the point with relative coordinates $[x_0, .. x_\nu]$ will be the point where the center of mass will go if each weight γ_i is replaced by $\gamma_i x_i$.

In particular, when u is actually the barycenter of the simplex (whose Cartesian coordinates are the arithmetic average of the Cartesian coordinates of the vertices), the weights γ_i are all equal, and the point with relative coordinates $[x_0, .. x_\nu]$ will be the center of mass of weights $x_0, .. x_\nu$ placed at the vertices of the simplex. See figure 12. This point also has the property that its distance to the face h^i of s opposite to vertex s^i, divided by the distance from s^i to h^i, is $x_i / \sum_j x_j$.

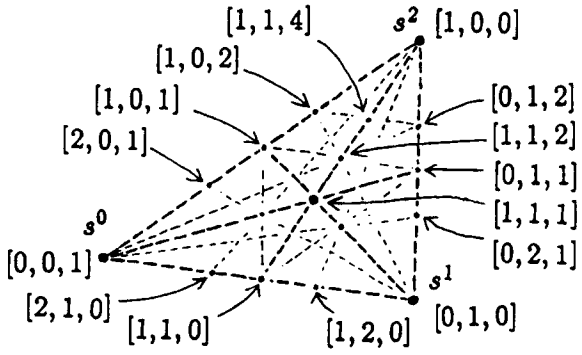

Figure 12. Barycentric coordinates.

We get this same coordinate system also when f is a mixed frame whose horizon is Ω. In either case, the resulting coordinates are called the *barycentric coordinates relative to the main simplex s.*

4.3. Uniqueness of projective maps

It is now time to prove that the objects we have been studying so far are indeed frames. We must verify that

Theorem 6. *Point frames and mixed frames are categorical.*

PROOF: Recall that an arrangement a is categorical if the only projective map on $Span(a)$ that takes a to itself is the identity. Let's first show that every standard point frame f of $\mathbf{T}_\nu$ is categorical.

So, suppose M is a map from $\mathbf{T}_\nu$ to $\mathbf{T}_\nu$ (or $\neg\mathbf{T}_\nu$) that takes the frame f to itself. In particular, M must take the canonical simplex of $\mathbf{T}_\nu$ to itself. It follows that its matrix must have positive coefficients in the main diagonal, and is zero everywhere else:

$$M = \begin{bmatrix} \gamma_0 & & 0 \\ & \ddots & \\ 0 & & \gamma_\nu \end{bmatrix} \tag{10}$$

Let σ be the signature of the frame f. If f is a point frame, then M must take the point $[\sigma] = [\sigma_0, .. \sigma_\nu]$ to itself. According to (10), $[\sigma]M = [\gamma_0\sigma_0 .. \gamma_\nu\sigma_\nu]$. It follows that $[\sigma]M = [\sigma]$ implies all γ_i must be equal.

Similarly, if f is a mixed frame M must keep fixed the hyperplane $\langle\sigma\rangle = \langle\sigma_0, .. \sigma_\nu\rangle$. The image of $\langle\sigma\rangle$ has coefficients $\langle\sigma\bar{M}\rangle$, where $\bar{M}$ is the adjoint of the matrix of M. The adjoint of (10) is

$$\bar{M} = \begin{pmatrix} \varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_\nu \end{pmatrix}$$

where $\varepsilon_i = \prod_{j \neq i} \gamma_j$. Therefore, $\langle\sigma\rangle M = \langle\varepsilon_0\sigma_0, .. \varepsilon_\nu\sigma_\nu\rangle$. In order to have $\langle\sigma\rangle M = \langle\sigma\rangle$ it is necessary that all ε_i, and therefore all γ_i, be the same.

In either case the matrix of M must be a positive multiple of the identity, and therefore M must be the identity map of $\mathbf{T}_\nu$. We conclude that the standard point frames and mixed frames are categorical. Because of theorems 2 and 5, it follows that all point frames and mixed frames are categorical.

QED.

5. Conclusions

To summarize this chapter, two point frames or two mixed frames f, g determine a projective map between their spanned spaces if and only if they have the same rank and signature.

In that case, the map is unique, and will take the oriented span of f to that of g. Said another way, if f and g are frames (positive or negative) for oriented flats R and S, then the two frames define a projective map from R to S if they have the same orientation, and a map from R to $\neg S$ if they have opposite orientations.

A projective frame for a space S also allows us to assign homogeneous coordinates to each point of S. The vertices of the main simplex of the frame get coordinates $[1, 0, 0, .. 0], [0, 1, 0, .. 0], \ldots, [0, 0, 0, .. 1]$. In the case of a point frame, its unit point gets coordinates $[\pm 1, \pm 1, .. \pm 1]$; in the case of a mixed frame, its horizon gets coefficients $\langle \pm 1, \pm 1, .. \pm 1\rangle$.

6. References

[1] H. Crapo and J. Ryan: **Scene analysis and geometric homology.** Proceedings of the 2nd ACM Symposium on Computational Geometry (June 1986), 125–132.

Chapter 13
Cross ratio

Measuring the length of a segment (or, more precisely, the ratio between two such lengths) is a fundamental operation of Euclidean geometry. Indeed, the Euclidean transformations — rigid motions and similarities — can be defined as the maps that preserve length, or the ratio of lengths. Indeed, the very word "geometry" reminds us that this whole area of mathematics was born as a science of measurement.

Given the obvious importance of this concept, it is natural to ask whether it has some analog in projective geometry. That is, can we define some notion of "length" or "length ratio" that is preserved by arbitrary projective maps? The answer is yes: such a notion can be defined, but it turns out to be somewhat more complicated than its Euclidean counterpart.

1. Cross ratio in unoriented geometry

It is not hard to show that that any function that depends on only two or three points and is invariant under arbitrary projective maps must be boolean- or sign-valued (e.g. "are the three points collinear?").

In classical projective geometry, one learns that the simplest real-valued invariant is the *cross ratio* of four collinear points, which can be defined as follows. If x, y, a, b are four distinct real numbers, then their cross ratio is the fraction

$$(x : y \mid a : b) \;=\; \frac{x-a}{b-x} \Big/ \frac{y-a}{b-y} \tag{1}$$

In general, if x, y, a, b are four distinct points on a line l, their cross ratio $(x : y \mid a : b)$ can be defined by picking an arbitrary Cartesian coordinate system on the line (i.e., an origin, a direction, and a unit of length), measuring the coordinate of each point

on this scale, and plugging those numbers into equation (1). See figure 1.

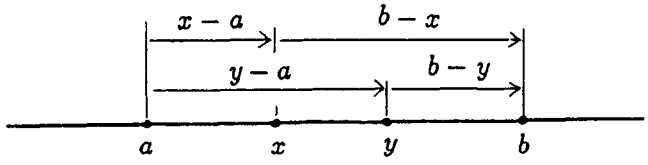

Figure 1. Cross ratio of four points.

It is easy to see that changes of coordinate system (translations, scalings, sign reversals) do not affect the result formula (1). It is only a little bit harder to show that formula (1) remains invariant when all four points are transformed by maps of the form $z \mapsto (\alpha z + \beta)/(\gamma z + \delta)$. These are precisely the projective maps of the real line. The invariance of the four-point cross ratio under projective maps follows from this observation. The assumption that all four points are distinct can be relaxed somewhat, particularly if we accept $1/0 = \infty = -\infty$ as a valid ratio. The definition can be extended also to the case where one of the points is at infinity. In particular, if a is zero and b goes to infinity, the cross ratio reduces to the plain ratio x/y. It is not worth going into details here, since we are going to do a similar analysis in section 2 for the two-sided version of cross ratio.

1.1. Interpreting the cross ratio

In order to gain a better intuition for the meaning of cross ratio, let's consider how the value of (1) varies when we keep a, y, b fixed, and move x along their common line, in the direction from a to b by way of y (possibly crossing $x = \infty$ along the way). See figure 2.

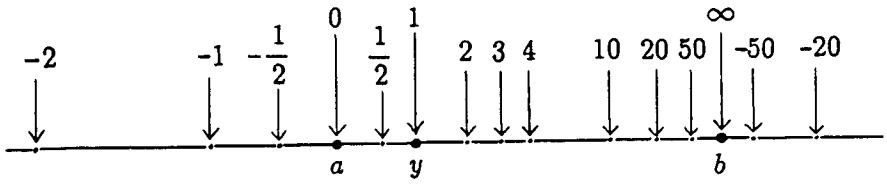

Figure 2. Values of $(x : y \mid a : b)$ as a function of x.

As x goes from a to b, the cross ratio spans the positive values from 0 (when $x = a$) to 1 (when $x = y$) to ∞ (when $x = b$). As x continues moving past b and goes back to a by the complementary route (the one not including y), the cross ratio goes

through the negative values, from $-\infty$ to 0. The point x where the cross ratio has value -1 is the *harmonic conjugate* of y with respect to a and b.

1.2. Computing the classical cross ratio

Recall that a point x of the unoriented projective line $\mathbf{P}_1$ with homogeneous coordinates $[x_0, x_1]$ corresponds to the point x_1/x_0 of the real line. Therefore, if we are given four distinct proper points x, y, a, b of $\mathbf{P}_1$, we can compute their classical cross ratio by the formula

$$
\begin{aligned}
(x : y \mid a : b) &= \frac{\left(\frac{x_1}{x_0} - \frac{a_1}{a_0}\right) \Big/ \left(\frac{b_1}{b_0} - \frac{x_1}{x_0}\right)}{\left(\frac{y_1}{y_0} - \frac{a_1}{a_0}\right) \Big/ \left(\frac{b_1}{b_0} - \frac{y_1}{y_0}\right)} \\
&= \frac{\frac{a_0x_1 - a_1x_0}{a_0x_0} \cdot \frac{y_0b_1 - y_1b_0}{y_0b_0}}{\frac{x_0b_1 - x_1b_0}{x_0b_0} \cdot \frac{a_0y_1 - a_1y_0}{a_0y_0}} \qquad (2) \\
&= \frac{\begin{vmatrix} x_0 & x_1 \\ a_0 & a_1 \end{vmatrix} \cdot \begin{vmatrix} b_0 & b_1 \\ y_0 & y_1 \end{vmatrix}}{\begin{vmatrix} x_0 & x_1 \\ b_0 & b_1 \end{vmatrix} \cdot \begin{vmatrix} a_0 & a_1 \\ y_0 & y_1 \end{vmatrix}}.
\end{aligned}
$$

If we view the cross ratio itself as a point on the projective line, its homogeneous coordinates are then

$$
(x : y \mid a : b) = \left[\begin{vmatrix} x_0 & x_1 \\ b_0 & b_1 \end{vmatrix} \cdot \begin{vmatrix} a_0 & a_1 \\ y_0 & y_1 \end{vmatrix}, \begin{vmatrix} x_0 & x_1 \\ a_0 & a_1 \end{vmatrix} \cdot \begin{vmatrix} b_0 & b_1 \\ y_0 & y_1 \end{vmatrix} \right] \qquad (3)
$$

or, in schematic form,

$$
(x : y \mid a : b) = \left[\begin{vmatrix} x \\ b \end{vmatrix} \cdot \begin{vmatrix} a \\ y \end{vmatrix}, \begin{vmatrix} x \\ a \end{vmatrix} \cdot \begin{vmatrix} b \\ y \end{vmatrix} \right] \qquad (4)
$$

2. Cross ratio in the oriented framework

I will take equations (3–4) as the *definition* of cross ratio on the oriented projective line $\mathbf{T}_1$. The cross ratio itself is to be viewed as a point of $\mathbf{T}_1$, so that a cross ratio of $3/2 = [2,3]$ is distinct from a cross ratio of $(-3)/(-2) = [-2,-3] = \neg[2,3]$. This means we must be doubly careful about the order of the four points in formulas (3–4).

2.1. Interpreting the cross ratio

To appreciate the meaning of formula (4), let's consider again what happens to the cross ratio when we move x and keep a, y, b fixed, as we did in section 1.1 for the unoriented version. One difference that comes up right away is that in the oriented world y may be in four projectively distinguishable positions with respect to a and b, and the cross ratio behaves differently in each case. So, let's first consider the case where y is on the open segment ab. See figure 3.

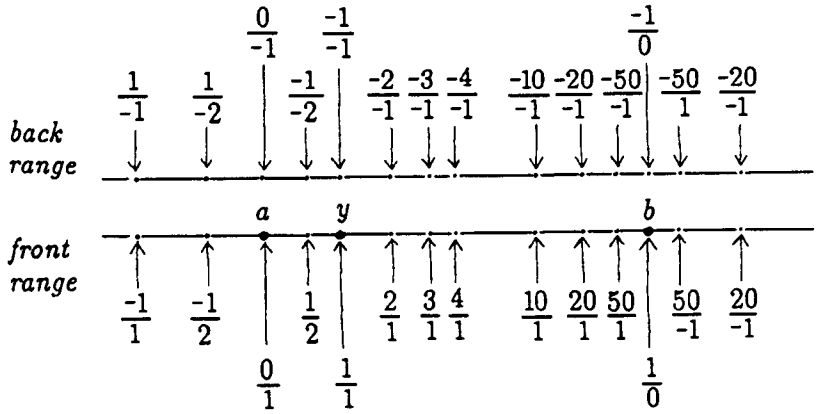

Figure 3. Values of $(x : y \mid a : b)$ as a function of x.

In that case, as x moves forward on the line $a \vee b$, the cross ratio $(x : y \mid a : b)$ moves forward on the line $\mathbf{T}_1$. Its value will be

$$\left\{\begin{array}{c}\text{front positive}\\ \text{back negative}\\ \text{back positive}\\ \text{front negative}\end{array}\right\} \quad \text{if } x \text{ is between} \quad \left\{\begin{array}{c}a \text{ and } b\\ b \text{ and } \neg a\\ \neg a \text{ and } \neg b\\ \neg b \text{ and } a\end{array}\right\} \tag{5}$$

In particular, as x moves from a to b along the segment ab, the cross ratio goes from $0 = [1,0]$ (when $x = a$) to $1 = [1,1]$ (when $x = y$) to $+\infty = [0,1]$ (when $x = b$).

In case y is not on the segment ab, we can understand what happens by noticing that if we replace any argument by its antipode, both coordinates of the

cross ratio are negated. That is,

$$\begin{aligned}(\neg x : y \mid a : b) &= (x : \neg y \mid a : b)\\ = (x : y \mid \neg a : b) &= (x : y \mid a : \neg b) = \neg(x : y \mid a : b).\end{aligned} \tag{6}$$

Therefore, if y is on the segment $b(\neg a)$, we have only to replace $(x : y \mid a : b)$ by $\neg(x : y \mid \neg a : b)$, and apply the above analysis. For example, we can say that as x moves from $\neg a$ to b the value of $(x : y \mid a : b)$ varies from $[-1, 0]$ through $[-1, -1]$ to $[0, -1]$.

2.2. Symmetry properties

The cross ratio has a number of symmetry properties which follow directly from the defining formulas. In particular, the cross ratio doesn't change if we swap the first pair of arguments with the second pair, or reverse the order of both pairs simultaneously. That is,

$$(x : y \mid a : b) = (a : b \mid x : y) = (y : x \mid b : a) = (b : a \mid y : x). \tag{7}$$

Also, reversing only the first (or last) pair has the effect of exchanging the two coordinates of the cross ratio. In the straight model of $\mathbf{T}_1$ (the two-sided real line) this is equivalent to taking the reciprocal. If we define $1/[r, s] = [s, r]$, we can write

$$(y : x \mid a : b) = (x : y \mid b : a) = 1/(x : y \mid a : b). \tag{8}$$

Swapping the innermost (or outermost) pair is numerically equivalent to computing one minus the original ratio, and moving to the antipodal range. More precisely, if we define $1 - [r, s] = [r, r - s]$, then

$$(x : a \mid y : b) = (b : y \mid a : x) = \neg\big(1 - (x : y \mid a : b)\big) \tag{9}$$

In all, the $4! = 24$ possible permutations split into six equivalence classes with four members each. If the original cross ratio is $\alpha = [r, s]$, the other five values that can be obtained this way are

$$\begin{aligned}(x : y \mid a : b) &= [r,\ s] &&= \alpha\\ (x : y \mid b : a) &= [s,\ r] &&= 1/\alpha\\ (x : a \mid y : b) &= [-r,\ s - r] &&= \neg(1 - \alpha)\\ (x : a \mid b : y) &= [s - r,\ -r] &&= \neg 1/(1 - \alpha)\\ (x : b \mid y : a) &= [-s,\ r - s] &&= \neg(1 - 1/\alpha)\\ (x : b \mid a : y) &= [r - s,\ -s] &&= \neg 1/(1 - 1/\alpha)\end{aligned} \tag{10}$$

2.3. Invariance under projective maps

Lemma 1. *For any points x, y, a, b of $\mathbf{T}_1$, and any projective map M of $\mathbf{T}_1$ to $\pm\mathbf{T}_1$, we have*

$$(xM : yM \mid aM : zM) = (x : y \mid a : b). \tag{11}$$

PROOF: Let M be the linear map of $\mathbf{R}^2$ to itself that induces M. From definition (4), we have

$$\begin{aligned}
(xM : yM \mid aM : bM) &= \left[\begin{vmatrix} xM \\ bM \end{vmatrix} \cdot \begin{vmatrix} aM \\ yM \end{vmatrix}, \begin{vmatrix} xM \\ aM \end{vmatrix} \cdot \begin{vmatrix} bM \\ yM \end{vmatrix} \right] \\
&= \left[\begin{vmatrix} x \\ b \end{vmatrix} \cdot \begin{vmatrix} a \\ y \end{vmatrix} \cdot |M|^2, \begin{vmatrix} x \\ a \end{vmatrix} \cdot \begin{vmatrix} b \\ y \end{vmatrix} \cdot |M|^2 \right] \qquad (12) \\
&= \left[\begin{vmatrix} x \\ b \end{vmatrix} \cdot \begin{vmatrix} a \\ y \end{vmatrix}, \begin{vmatrix} x \\ a \end{vmatrix} \cdot \begin{vmatrix} b \\ y \end{vmatrix} \right] \\
&= (x : y \mid a : b)
\end{aligned}$$

QED.

This result makes it possible to compute the cross ratio of four points x, y, a, b lying on an arbitrary one-dimensional space l. If φ is any isomorphism from l to $\mathbf{T}_1$, the cross ratio $(x : y \mid a : b)$ is by definition the same as $(x\varphi : y\varphi \mid a\varphi : b\varphi)$, computed according to equation (4). Lemma 1 above assures us that the cross ratio does not depend on which isomorphism we use: if φ, η are two isomorphisms from l to $\mathbf{T}_1$, then $\overleftarrow{\varphi}\eta$ is a projective map of $\mathbf{T}_1$ to itself, which means $(x\varphi : y\varphi \mid a\varphi : b\varphi) = (x\eta : y\eta \mid a\eta : b\eta)$. By the same argument, we can generalize lemma 1 to any projective line; that is,

Theorem 2. *The cross ratio of four collinear points is invariant under arbitrary projective maps.*

2.4. Cross ratio as relative coordinates

Let x, y, a, b be four collinear points, with y on the segment ab. Viewed as a function of x, the cross ratio $(x : y \mid a : b)$ is a map that takes a to $[1,0]$, y to $[1,1]$, and b to $[0,1]$. By inspecting formula (4), we can see that, if the other points are fixed, the coordinates of the cross ratio are homogeneous linear functions of those of x. This function is therefore the map that takes the point frame (a, b, y) to the standard point frame of $\mathbf{T}_1$ with same signature $(++)$. In other words, *the cross ratio* $(x : y \mid a : b)$ *gives the homogeneous coordinates of* x *relative to the point frame* (a, b, y).

Unfortunately, this is true only if y lies on the segment ab. In general, if we compute the coordinates of x relative to (a, b, y) by the formulas given in the previous chapter, we will get

$$\operatorname{sign}\begin{vmatrix} a \\ b \end{vmatrix} \circ \left[\begin{vmatrix} x \\ b \end{vmatrix} \cdot \operatorname{abs}\begin{vmatrix} a \\ y \end{vmatrix}, \begin{vmatrix} a \\ x \end{vmatrix} \cdot \operatorname{abs}\begin{vmatrix} y \\ b \end{vmatrix} \right]. \tag{13}$$

Comparing this with formula (4), we can see that the two agree only when

$$\operatorname{sign}\begin{vmatrix} a \\ b \end{vmatrix} = \operatorname{sign}\begin{vmatrix} a \\ y \end{vmatrix} = \operatorname{sign}\begin{vmatrix} y \\ b \end{vmatrix}, \tag{14}$$

that is, when y is in the segment ab. The difference between the two formulas arises because of the decision we made in chapter 12 to use the canonical simplex of $\mathbf{T}_\nu$ as the the main simplex of all standard frames. As a consequence, the formulas for coordinates relative to a frame $f = (s^0, \ldots s^\nu, u)$ treat the unit point u differently from the other points. Replacing u by $\neg u$ in f has no effect in the map M_f, whereas replacing one of the s^i by $\neg s^i$ modifies the map so as to send $\mathbf{e}^i$ to $\neg s^i$ instead of s^i.

We could have removed the discrepancy by taking equation (13) instead of (4) as the definition of cross ratio. However, the "absolute cross ratio" we would get this way would not have the nice symmetry properties of the classical cross-ratio. Alternatively, we could have defined standard frames in such a way that the formulas for relative coordinates were equally sensitive to the orientation of all points of the frame, and reduced to formula (4) in the one-dimensional case. The symmetrical cross ratio would come out as a special case of relative coordinates. This would have the disadvantage of making relative coordinates less predictable. The issue boils down to which is more important in practice, symmetry or predictability. As I lack the necessary experience, I will leave the issue open, without trying to unify the two concepts.

Chapter 14
Convexity

Convexity theory provides an important example of the advantages of the two-sided approach. In affine geometry, a figure is said to be convex if it contains every segment whose endpoints lie in the figure. This concept has no clean counterpart in classical projective geometry, essentially because one cannot define unambiguously *the* segment connecting two given points. By contrast, in two-sided geometry the segment pq is well defined and unique, as long as the two points are not antipodal. Moreover, it is a purely projective notion, that can be defined in terms of join. Therefore, in two-sided geometry we can define a notion of convexity that is preserved by projective maps, and yet preserves most of the properties of affine convexity. This allows us to consistently apply the tools of projective geometry, and in particular projective duality, to the development of theorems and algorithms involving convexity. The result is a theory of convex sets that is cleaner and richer than the affine version.

1. Convexity in classical projective space

Attempts to extend the notion of convexity to *unoriented* projective space have followed two major approaches. The first approach is to assign special meaning to some fixed line, say the line at infinity Ω. A convex set is then defined as one that avoids Ω, and that contains every line segment whose ends lie in the set and which does not cross Ω. The problem with this solution is that is doesn't give us anything new: by excluding Ω we are simply working on the affine plane, and losing most of the advantages of projective geometry. For one thing, convexity will be preserved only by those projective maps that take Ω to itself, i.e. the affine maps.

A second approach, due to Sylvester, is to say that a set X is convex if there is *some* line l disjoint from X such every line segment with ends in X that does not cross l is contained in X. The difference with the previous approach is that the line l is allowed to depend on X. This brand of convexity is preserved by projective maps. Unfortunately, it lacks other properties we are accustomed to associate with convexity, such as closure under intersection. In fact, the intersection of two convex

sets may consist of two separate components. See figure 1.

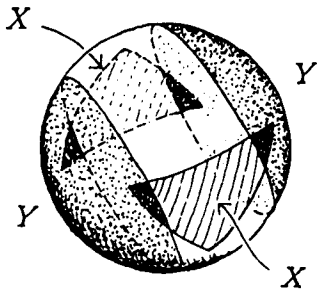

Figure 1. Intersecting convex sets in the classical projective plane.

2. Convexity in oriented projective spaces

In *oriented* projective geometry we can use the affine definition of convexity almost without change. Essentially, we can define a convex set as one which contains every segment whose endpoints lie on the set. However, before we use this definition we have to clarify the meaning of "segment," especially in the case of antipodal points.

2.1. Open and closed segments

Recall that in section 4:3.1 we defined the open segment pq, for two distinct and non-antipodal points of $\mathbf{T}_\nu$, as the set of all points x such that $(p;x)$ and $(x;q)$ are proper simplices equivalent to $(p;q)$. This condition is equivalent to $p \vee r = r \vee q = p \vee q \neq \mathbf{0}$. For the purposes of the present chapter, it is convenient to extend this definition by letting the open segment pp to be the empty set, and the open segment $p(\neg p)$ consist of a single element, the non-point $\mathbf{0}^1$. It is also convenient to let the open segment $p\mathbf{0} = \mathbf{0}p$ be the empty set.

By definition, the *closed segment* pq is the open segment pq together with the points p and q. If p and q are distinct and not antipodal, the closed segment pq is the shorter of the two closed arcs of great circle connecting them. If $p = q$ we get a single point, if $p = \neg q$ we get the two points plus the null object $\mathbf{0}^1$, and if $p = \mathbf{0}$ or $q = \mathbf{0}$ we get only the two points.

2.2. Convex sets

Now that we know what a segment is, let's go back to the definition of convex set. What happens if the set contains pairs of antipodal points? By our definition, the set must also contain the non-point $\mathbf{0}$. We can either forbid such pairs (on the grounds that the segment connecting them includes the non-point $\mathbf{0}$), or accept them

(since all *points* on the segment pq are in the set). These two alternatives lead to two different notions of convexity:

Definition 1. A set X of points is *quasi-convex* if for any two non-antipodal points p, q in X the segment pq is also in X.

Definition 2. A set X of points is *strictly convex* if it is nonempty, includes no antipodal pairs, and for any two points p, q in X the segment pq is also in X.

In other words, a quasi-convex set is strictly convex iff it is non-empty and contains no antipodal pair of points. Note that if X is contained in the front range of $\mathbf{T}_\nu$, the two definitions agree, and describe precisely those subsets of the front range that are convex in the sense of affine geometry.

Examples of strictly convex subsets of $\mathbf{T}_2$ are a single point, the interior of a proper simplex, an open half-space, an open wedge (the intersection of two half-spaces), and half of a straight line with at most one of its endpoints. In particular, any affinely convex subset of the front or back ranges of the straight model is a strictly convex set by definition 2. The converse is not true, however: as figure 2 shows, a strictly convex subset of $\mathbf{T}_\nu$ may extend across both ranges, and look rather non-convex at first glance.

Figure 2. A strictly convex subset of $\mathbf{T}_2$, in the straight model.

Some examples of quasi-convex sets that are not strictly convex are: a straight line, a closed half-space, a pair of antipodal points, the whole two-sided space, and one half of a straight line with both its endpoints. Note that there is no quasi-convex set that is bigger than the latter and smaller than a whole line: in terms of the spherical model, if a quasi-convex set includes more than the closed half of a great circle, it contains the whole great circle.

3. Properties of convex sets

3.1. Invariance under projective maps

Note that since segments can be defined in terms of join, they are defined on arbitrary two-sided spaces, not just on the standard space $\mathbf{T}_\nu$. Therefore, the same observation applies to the notions of quasi-convexity and strict convexity. Furthermore, since isomorphisms by definition commute with the join operation, they must take segments to segments. We can immediately conclude that

Theorem 1. *Quasi-convexity and strict convexity are preserved by isomorphisms between projective spaces.*

In particular, these notions are preserved by projective maps of $\mathbf{T}_\nu$ to itself. Also, theorem 1 implies that any projective properties of convex subsets that we can prove on the standard two-sided space $\mathbf{T}_\nu$ are valid also on any two-sided space of same dimension.

3.2. Intersection properties

Theorem 2. *Quasi-convex sets are closed under arbitrary intersections.*

PROOF: Consider an arbitrary, non-empty family of quasi-convex subsets of $\mathbf{T}_\nu$, and let X be its intersection. Let p, q be any two distinct, non-antipodal points of X. The points p and q must be in every member of the family; by definition every member must contain the segment pq, which therefore must be contained in X.

QED.

The following are obvious corollaries:

Theorem 3. *Strictly convex sets are closed under arbitrary intersections, as long as the intersection is non-empty.*

Theorem 4. *A subset X of $\mathbf{T}_\nu$ is quasi-convex if and only if its intersection with every line of $\mathbf{T}_2$ is quasi-convex.*

PROOF: The "only if" part is a corollary of theorem 2, since a line of $\mathbf{T}_2$ (considered as the set of points) is quasi-convex.

As for the converse, suppose X is not quasi-convex. Then there are distinct, non-antipodal points p, q in X such that the segment pq is not contained in X. Let m be the line $p \vee q$; the points p and q are in $X \cap m$, but the segment pq is

not. Therefore, if X is not quasi-convex, there is some line m such that $X \cap m$ is not quasi-convex.

QED.

Theorem 5. *A subset X of a space S is strictly convex if and only if it is non-empty and its intersection with every line of S is either empty or strictly convex.*

Note that a strictly convex subset of a line has one of the following forms: a point; a proper line segment with zero, one, or two of its endpoints; or a half-line with at most one of its endpoints.

3.3. The interior of convex sets

A useful property of convex sets is the following:

Theorem 6. *In a space S with positive rank, a quasi-convex subset has empty interior if and only if it is contained in some hyperplane of S.*

PROOF: The "if" part is trivial, since in spaces of positive rank hyperplanes have no interior points.

For the "only if" part, observe that a subset X of $\mathbf{T}_\nu$ that is not contained in any hyperplane must include n points that are linearly independent when viewed as vectors of $\mathbf{R}^n$; that is, it must include a proper ν-dimensional simplex. If $n \geq 2$, the points inside that simplex (with signature $++\cdots+$, as defined in chapter 4) are easily shown to be interior points of X, if X is quasi-convex. If $n = 1$, the simplex has one vertex, and no points are inside it. However, in that case the space $\mathbf{T}_\nu = \mathbf{T}_0$ is discrete (each point is both an open and a closed set), so every point of X is interior. In any case, this shows that a quasi-convex set not contained in any hyperplane must have some interior points.

QED.

Theorem 7. *In a space of positive rank, a quasi-convex subset X with nonempty interior is contained in the closure of its interior.*

PROOF: The case when S has rank one is trivial, so let's suppose $\text{rank}(S) = n \geq 2$. If X has non-empty interior, it is not contained in any hyperplane of S. Therefore, for any point $x \in X$ there must be $n-1$ other points that together with x form a a proper $(n-1)$-dimensional simplex. The point x is in the closure of this simplex. The interior of this simplex is contained in the interior of X, therefore the closure of the simplex is contained in the closure of the interior of X. Since this holds for all $x \in X$, the theorem is proved.

QED.

Theorem 8. *Any open subset of a space S that is proper, non-empty, and quasi-convex is strictly convex.*

PROOF: Let X be such a set, and suppose it contained two antipodal points $p, \neg p$. Since X is open, p must lie in some open neighborhood $N \subseteq X$. Let l be any line through p. The line l includes $\neg p, p$, and points of $N \cap l$ (an open subset of l) that approach p on both sides. Since X is quasi-convex, it follows that $l \cap X$ must be the whole l. Since this holds for all lines through p, X must be the whole space, a contradiction. We conclude that X has no antipodal points, and therefore is strictly convex.

QED.

Informally, theorems 7 and 8 say that quasi-convex sets that are not entirely flat don't have any flat appendages: they consist of an open strictly convex set and some points on its boundary.

3.4. Analytic characterization of convexity

Analytically, a point r is on the closed segment pq if and only if its *homogeneous* coordinates linearly interpolate between those of p and q. If $p = [x]$, $q = [y]$, and $r = [z]$, this condition means $z_i = \alpha x_i + (1 - \alpha) y_i$, for all i and for some α with $0 \leq \alpha \leq 1$. Since positive common factors can be ignored in homogeneous coordinates, this is equivalent to there being $(\alpha, \beta) \neq (0, 0)$ and $z_i = \alpha x_i + \beta y_i$ for all i. In fact, we can conclude that r is on the closed segment pq if and only if $r = p$, $r = q$, or there are $x, y \in \mathbf{R}^n$ such that $p = [x]$, $q = [y]$, and $r = [x + y]$. It follows that

Theorem 9. *A subset X of $\mathbf{T}_\nu$ is quasi-convex if and only if it contains the point $[x + y]$, for any distinct and non-antipodal points $[x], [y] \in X$.*

In the spherical model, we can see that a closed segment of $\mathbf{R}^n$ that does not contain the origin $\vec{0} = (0, .. 0)$ is mapped by central projection onto a segment of the two-sided plane. This gives us two additional characterizations of convex sets:

Theorem 10. *A subset X of $\mathbf{T}_\nu$ is quasi-convex if and only if it is the central projection on $\mathbf{S}_\nu$ of $Y \setminus \vec{0}$, where Y is a convex subset of $\mathbf{R}^n$.*

PROOF: If Y is a convex subset of $\mathbf{R}^n$, the quasi-convexity of its projection X follows from the definition and from the observations in the preceding paragraph.

Conversely, let X be a quasi-convex subset of $\mathbf{T}_\nu$. Let's consider X as a subset of the unit sphere of $\mathbf{R}^n$, and let Y be the union of all closed segments connecting the origin of $\mathbf{R}^n$ to a point of X. See figure 3. It is easy to check that

Y is convex, and obviously the central projection of $Y \setminus \vec{0}$ on $\mathbf{S}_2$ is X.

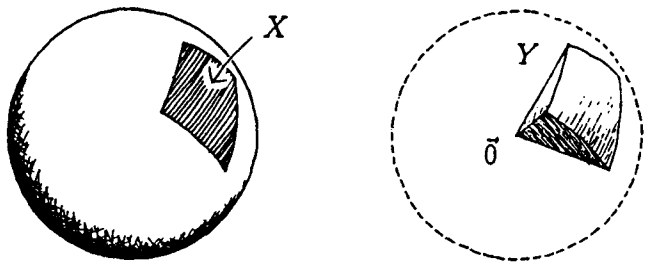

Figure 3. Convexity in $\mathbf{T}_\nu$ and in $\mathbf{R}^n$.

QED.

Theorem 11. *A subset X of $\mathbf{T}_\nu$ is strictly convex if and only if it is the central projection of a convex subset of $\mathbf{R}^n \setminus \vec{0}$.*

3.5. Convexity and projective maps

Among the many trivial corollaries of theorems 10 and 11, we have:

Theorem 12. *If X is a quasi-convex set of points and F is any projective function defined on X, then $F(X \setminus Null(F))$ is quasi-convex.*

PROOF: Let X be a quasi-convex subset of $\mathbf{T}_\nu$, and F a projective function of $\mathbf{T}_\nu$ to $\mathbf{T}_\kappa$. Following theorem 10, let Y be a convex subset of $\mathbf{R}^n$ such that X is the central projection of $Y \setminus \vec{0}$. Let F be a function of $\mathbf{R}^n$ to $\mathbf{R}^k$ that generates the projective function F. Then $F(X \setminus Null(F))$ is the central projection of $F((Y \setminus \vec{0}) \setminus Null(F)) = F(Y \setminus Null(F))$. By the definition of null space, $F(Y \setminus Null(F)) = F(Y) \setminus \vec{0}$. Since linear functions preserve convexity in $\mathbf{R}^n$, $F(Y)$ is convex. By theorem 10, it follows that X is quasi-convex.

QED.

Theorem 13. *If F is a projective function, and X is a strictly convex set disjoint from $Null(F)$, then $F(X)$ is strictly convex.*

PROOF: Following theorem 11, let $X \subseteq \mathbf{T}_\nu$ be the central projection of $Y \subseteq \mathbf{R}^n \setminus \vec{0}$. Let $F = [\![F]\!]$ where F is a linear function from $\mathbf{R}^n$ into $\mathbf{R}^k$. Then $F(X)$ is the central projection of $F(Y)$. From $X \cap Null(F) = \emptyset$ we have $Y \cap Null(F) = \emptyset$, and therefore $F(Y) \subseteq \mathbf{R}^k \setminus \vec{0}$. By theorem 11, $F(X)$ is strictly convex.

QED.

4. The half-space property

The set of all points in a two-sided space is quasi-convex, but not strictly convex. How big can a strictly convex set be? Roughly speaking, not bigger than half of the containing space. In this section we will make this statement more precise.

4.1. Supporting half-spaces

Lemma 14. *Every open or closed set of points that is strictly convex is contained in the positive side of some hyperplane.*

PROOF: Let X be an open strictly convex subset of $\mathbf{T}_\nu$. We will prove the theorem by induction on ν. If $\nu = -1$ the theorem is trivially true: $\mathbf{T}_{-1}$ has no points, hence no strictly convex subsets (its only flats are the vacua, which are also the universes). If $\nu = 0$, there are only two points, the universe Υ and its opposite. In that case we must have $X = \{\Upsilon\}$ or $X = \{\neg\Upsilon\}$. Then the hyperplane Λ or $\neg\Lambda$, respectively, will leave X on its positive side.

Now suppose $\nu \geq 1$. If X is strictly convex, it must be a proper subset of $\mathbf{T}_\nu$, and disjoint from its own antipodal image $\neg X$. Its set-theoretic complement $\mathbf{T}_\nu \setminus X$ must be a proper nonempty subset of $\mathbf{T}_\nu$. Since $\mathbf{T}_\nu$ is connected, X and $\mathbf{T}_\nu \setminus X$ cannot be both open or both closed. On the other hand, the map $x \mapsto \neg x$ is continuous and one-to-one, so X and $\neg X$ are both open or both closed. We conclude that $\neg X$ is a proper subset of $\mathbf{T}_\nu \setminus X$. That is, there is some point p that is neither in X nor in $\neg X$. Obviously, the same is true of $\neg p$.

Now let π be a right complement of p in $\mathbf{T}_\nu$, that is, a hyperplane such that $p \vee \pi = \Upsilon$. Let F be the projection of $\mathbf{T}_\nu$ from p onto π:

$$F(x) = (p \vee x) \wedge \pi \tag{1}$$

See figure 4.

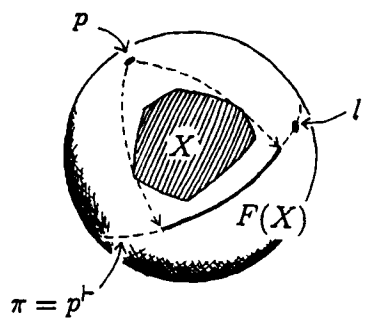

Figure 4.

The null space of this function is the set $\{p, \neg p\}$, and its range is the whole hyperplane π. Since $X \subseteq \mathbf{T}_\nu \setminus Null(F)$, by theorems 11:1 and 11:2 $F(X)$ is open if X is open, and closed if X is closed. According to theorem 13, $F(X)$ is a strictly convex subset of π. By induction, there must be a hyperplane l of π that leaves $F(X)$ on its positive side. That is, for all $x \in X$ we must have $F(x) \vee l = \pi$. Therefore $p \vee F(x) \vee l = p \vee \pi = \Upsilon_\nu$. Since $p \vee F(x) = p \vee x$, we have $p \vee x \vee l = \Upsilon_\nu$, which implies $x \vee \neg(p \vee l) = \Upsilon_\nu$. Since this holds for all $x \in X$, we conclude that X lies in the positive side of the hyperplane $\neg(p \vee l)$. QED.

4.2. Perfect half-spaces

Note that theorem 14 applies only to strictly convex sets that are either open or closed. For general strictly convex sets the theorem is false: a counterexample is the subset of $\mathbf{T}_1$ consisting of one half circle plus one of its endpoints. Fortunately, we can fix this problem quite easily, as follows.

Let us define a *perfect half* of a space S recursively as being

- the empty set, if $\text{rank}(S) = 0$, or
- an open half-space of S, plus a perfect half of its bounding hyperplane, if $\text{rank}(S) > 0$.

For example, a *perfect half-line* consist of an open half-line plus exactly one of its endpoints. A *perfect half-plane* of $\mathbf{T}_2$ consists of an open hemisphere of $\mathbf{S}_2$, an open half-line on its boundary, and one endpoint of that half-line (figure 5).

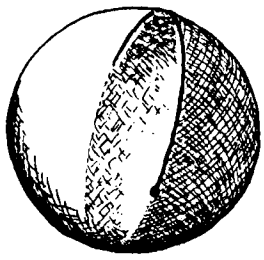

Figure 5. A perfect half-plane of $\mathbf{T}_2$.

The perfect halves of a space S are all projectively equivalent. However, if $\text{rank}(S) \geq 1$ they can be divided in two classes according to their "handedness," that is, according to whether the map relating them is positive or negative. For example, we can classify the perfect half-lines of $\mathbf{T}_1$ according to whether they are open at the

"forward" or at the "backward" end, as seen from any interior point. Note that the antipodal image of a perfect half H of S is another perfect half of S (with same or opposite handedness, depending on whether S has even or odd rank), and is exactly the same as the set-theoretic complement $S \setminus H$.

It is not hard to prove recursively that the intersection of a perfect half-space of S with any flat a of S is a perfect half of a. It is easy to see that perfect half-lines are strictly convex. Because of theorem 5, it follows that any perfect half-space is strictly convex. Since a perfect half-space includes one member of every antipodal pair, it cannot be augmented without ceasing to be convex. In other words, perfect half-spaces are *maximal strictly convex sets.* The converse is also true: every maximal strictly convex subset of a space S is a perfect half of S. This is a trivial consequence of the following theorem:

Theorem 15. *Every strictly convex subset of a space S is contained in a perfect half of S.*

PROOF: The proof is by induction on the rank of S. If S has rank zero, it has no strictly convex subsets, and the theorem is vacuously true. So, assume S has positive rank, and X is a strictly convex subset.

If X has no interior points, then by theorem 6 it is contained in some hyperplane f. If X has some interior points, then its interior is a strictly convex open set. By theorem 14, the interior of X is contained in the positive side of some hyperplane f. By theorem 7, X is contained in the closure of the interior of X, and hence it is contained in the closure of f's positive side.

In either case we conclude that X is contained in the union of f and the positive side of f. If $X \cap f$ is not empty, it is a strictly convex subset of f; by induction, $X \cap f$ is contained in some perfect half H of f. If $X \cap f$ is empty, this is trivially true. Therefore, X is contained in the perfect half of S consisting of H and the positive side of f.

QED.

In fact, we can prove a much stronger result:

Theorem 16. *Any strictly convex subset X of a space S is the intersection of the perfect half-spaces of S that contain X.*

PROOF: By theorem 15, there is at least one perfect half-space containing X, so the intersection Y of all such half-spaces is well-defined. Obviously, $X \subseteq Y$. To show that $Y \subseteq X$, we have to prove that for every point $p \notin X$, there is a perfect half-space that contains X but not p.

Consider such a point p. If $\neg p \in X$, then any perfect half-space that includes X automatically excludes p, and we are done. If $\neg p \notin X$, then by theorem 13 the

projection of X onto any hyperplane h complementary to p is a strictly convex subset of h. By theorem 15, it is contained in some perfect half H of h. Let Y be the set of all points of S that project onto H, plus the point $\neg p$. Obviously Y contains X but not p; it is easy to check that Y is a perfect half of S.

QED.

Chapter 15
Affine geometry

Oriented projective geometry is able to simulate all the constructions and algorithms of affine, Euclidean and Cartesian geometry. We have only to think of the straight model of $\mathbf{T}_\nu$, and restrict our attention to its front range. A Euclidean flat is simulated by the front part of a flat of $\mathbf{T}_\nu$; the projective maps that take the front range to itself provide all Euclidean and affine transformations; and so on. In this and the following chapters we will look into this emulation in more detail.

1. Cartesian coordinates

As we recall, the analytic and straight models of $\mathbf{T}_\nu$ are related by central projection

$$[x_0, x_1, .. x_\nu] \quad \mapsto \quad \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \cdots, \frac{x_\nu}{x_0}\right) \tag{1}$$

By definition, the projected point lies in front or back range, depending on whether x_0 is positive or negative; if x_0 is zero, the projection is the point at infinity in the direction $(x_1, x_2, .. x_\nu)$. This map is the standard way of converting from homogeneous to Cartesian coordinates. The inverse map is

$$(x_1, x_2, .. x_\nu) \quad \mapsto \quad [\pm 1, x_1, x_2, .. x_\nu], \tag{2}$$

where the weight coordinate is $+1$ for points on the front range, and -1 for points on the back.

In principle, formulas (1) and (2) are all we need to emulate Cartesian (hence Euclidean) geometry in the two-sided framework. However, as the next few sections will make clear, there are a few subtle points in the handling of signs which require some careful thought.

1.1. The midpoint of a segment

For example, let $p = [u, a, b]$ and $q = [v, c, d]$ be points on the front range of $\mathbf{T}_2$. The midpoint of the segment pq has Cartesian coordinates

$$\frac{1}{2}\left(\frac{a}{u} + \frac{c}{v},\ \frac{b}{u} + \frac{d}{v}\right) = \frac{1}{2uv}(va + uc,\ vb + ud) \tag{3}$$

Converting back to homogeneous coordinates, we finally get

$$\mathrm{mid}([u, a, b],\ [v, c, d]) \ = \ [\,2uv,\ va + uc,\ vb + ud\,]. \tag{4}$$

1.2. Natural or absolute?

This derivation is correct as long as p and q lie on the front range. When this is not the case, however, formula (4) produces unexpected results. Note that formula (4) is a bilinear function of the coordinates of p and q: therefore, when p or q is replaced by its antipode, the same happens to their "midpoint." So, if p and q are both on the back range, point (4) lies on the *front* range, and indeed is the midpoint of $\neg p$ and $\neg q$. Worse still, when p and q are on opposite ranges, (4) gives a point on the back range that is the midpoint of either p and $\neg q$, or of $\neg p$ and q.

If we don't like this behavior, we can use instead of (4) the formula

$$\mathrm{mid}(\ [u, a, b],\ [v, c, d]\ ) \ = \ \left[\ |v|\,u + |u|\,v,\ \ |v|\,a + |u|\,c,\ \ |v|\,b + |u|\,d\ \right]. \tag{5}$$

This gives always a point on the segment pq (provided p and q are not both at infinity, and $p \neq \neg q$). When p and q are on the same range, the result is the midpoint of pq. When p and q are on opposite ranges, the result is the point where that segment crosses the line at infinity.

The above situation is similar to the one we encoutered while defining cross-ratio in chapter 13, and occurs over and over again in the derivation of formulas for two-sided geometry. A straightforward way of deriving such formulas is to compute the Cartesian coordinates of the operands, apply the appropriate Cartesian geometry formulas, and convert the result back to homogeneous coordinates, with any divisions eliminated by a suitable rescaling. However, the "natural" formulas we usually obtain through this route give the intuitively expected result only for points on the front range. It is often possible to modify these formulas (usually, by adding some absolute-value operations in the right places) so that they retain their intuitive meaning over a wider range of arguments. One disadvantage of these "absolute" formulas is that they are not linear in the coordinates of the operands, and therefore the result changes in a complicated way when the operands' orientations are reversed. Only

experience will tell which flavor of formula is best for practical use, or whether both should be provided.

1.3. Why homogeneous coordinates?

The use of homogeneous coordinates for affine and Euclidean geometry, when plain Cartesian coordinates are enough, may seem to some readers a mathematical overkill and a waste of resources. One reason for doing so is the ability to handle degenerate situations, such as the meet of parallel lines. This freedom to leave the confines of the Euclidean plane in intermediate computations often simplifies the programs enormously, even if the input data and the final results have to be represented in Cartesian coordinates. Another reason is standardization: life is much simpler if all geometric software in a computer system uses the same data format. Finally, with homogeneous coordinates we are often able to eliminate the need for division operations, except at the very end when converting the results back to Cartesian coordinates.

2. Two-sided affine spaces

Let's now consider in particular how we can emulate affine geometry in the two-sided framework. The notions of affine geometry include *direction*, *parallelism*, *affine map*, *affine ratio*, *midpoint*, *barycenter*, and many others. They can all be defined in terms of meet and join, if we let some distinguished hyperplane play a special role in the definitions. This motivates the following definitions:

Definition 1. A *two-sided affine space* is a pair $A = (S, h)$ where S is an oriented projective space and h one of its hyperplanes, the *horizon* of A.

Definition 2. The *canonical two-sided affine space* $\mathbf{A}_\nu$ consists of the space $\mathbf{T}_\nu$, with Ω_ν as the horizon.

Recall that the hyperplane Ω of $\mathbf{T}_\nu$, was defined as the flat generated by the last ν points $(\mathbf{e}^1; .. \mathbf{e}^\nu)$ of the standard simplex. It follows immediately that the point $\mathbf{e}^0$ lies on the positive side of Ω. In the straight model of $\mathbf{T}_\nu$, Ω is the hyperplane at infinity oriented so that its positive side is the front range of $\mathbf{T}_\nu$. In the case of $\mathbf{T}_2$, Ω is the line at infinity that turns counterclockwise as seen from the origin. In the case of $\mathbf{T}_3$, Ω is the plane at infinity, oriented so that its circular arrow turns *clockwise* as seen from the origin. The flats of $\mathbf{T}_\nu$ contained in Ω are called *improper*, and all the others *proper*. By this definition, the universe $\mathbf{T}_\nu$ is a proper flat, and the vacuum is an improper one.

In what follows, by *affine space* I will usually mean a two-sided one. When

necessary, I will say the *one-sided affine space* to refer to the classical one.

2.1. Directions

If a is a proper flat, then $a \wedge \Omega$ is always defined, and is a flat at infinity with rank one less than the rank of a. I will call this the *direction of* a, and denote it by $\operatorname{dir} a$. Note that $\operatorname{dir}(\neg a) = \neg(\operatorname{dir} a)$.

For example, the direction of a proper line is the point of Ω where it exits the front half of $\mathbf{T}_\nu$. The direction of a proper plane is one of the two lines at infinity that lie on that plane, namely the one which turns around the front part of the plane in a way that agrees with the orientation of a. See figure 1.

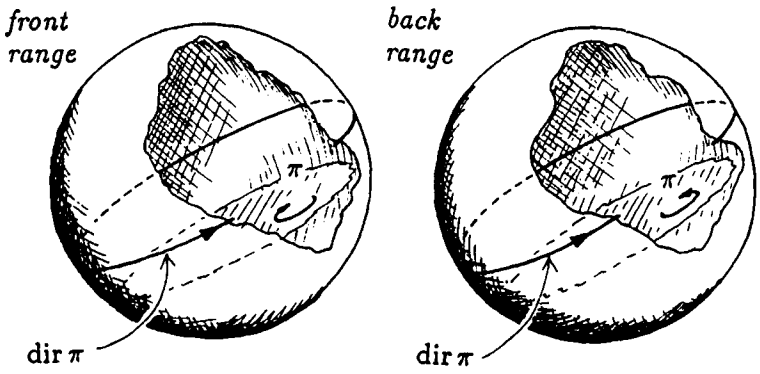

Figure 1. Direction of a plane of $\mathbf{T}_3$.

The hyperplane Ω cuts every proper flat a in two open subsets, one in each range of $\mathbf{T}_\nu$. A point p will be in the front part of a if and only if $p \vee \operatorname{dir} a = a$. In particular, a point p is in the front range of $\mathbf{T}_\nu$ if and only if $p \diamond \Omega = +1$.

As for the trivial cases, the direction of a point is Λ or $\neg\Lambda$ depending on whether the point lies on the front or back range of $\mathbf{T}_\nu$; and the direction of $\mathbf{T}_\nu$ is Ω itself.

2.2. Parallelism

A flat a is *parallel* to a flat b if $\operatorname{dir} a \subseteq \operatorname{dir} b$, of $\operatorname{dir} b \subseteq \operatorname{dir} a$; I will denote this by $a \parallel b$. Note that the flats have to be proper. In particular, every flat is parallel to itself and to its opposite, and to any sub- or super flat. A proper point is parallel to every proper flat, and every proper flat is parallel to the universe Υ_ν.

Note that according to this definition the parallel predicate is symmetric but in general not transitive: if a line l is parallel to a plane π, and π is parallel to another line m, it doesn't follow that $l \parallel m$. However, from $a \parallel b$ and $b \parallel c$ we can deduce $a \parallel c$, if $\operatorname{rank}(a) \geq \operatorname{rank}(b) \geq \operatorname{rank}(c)$ or $\operatorname{rank}(a) \leq \operatorname{rank}(b) \leq \operatorname{rank}(c)$.

If two flats a, b have the same rank k, their directions have same rank $k-1$. In this situation we can distinguish the case where a and b are *co-parallel* ($\mathrm{dir}\, a = \mathrm{dir}\, b$) from the case where they are *contra-parallel* ($\mathrm{dir}\, a = \neg\, \mathrm{dir}\, b$). I will denote these cases by $a \upuparrows b$ and $a \upharpoonleft\downharpoonright b$, respectively.

If f is a proper flat of rank r, and p is any point on the front range, then $p \vee \mathrm{dir}\, f = p \vee (f \wedge \Omega)$ is the (unique) flat with rank r that is co-parallel to f and passes through p. This is the oriented version of Euclid's Fifth Postulate: "through a point not on a line there is a unique parallel to that line." (If p lies on the back range, this formula gives a flat contra-parallel to f).

2.3. Affine spaces and subspaces

The notion of direction and the predicates $\|$, $\upuparrows$, and $\upharpoonleft\downharpoonright$ can be defined relative to an arbitrary affine space $A = (S, h)$ in the obvious way. The direction of a sub-flat f of S that is not contained in h is $f \wedge_S h$, and two flats a, b in S are co-parallel iff $a \wedge_S h = b \wedge_S h$. The *front range* of A is the positive side of h relative to S, that is, the set of points p such that $p \vee h = S$. Their antipodals constitute the *back range* of A.

If $A = (S, h)$ is an affine space, and f is a flat contained in S but not in h, then $F = (f, f \wedge_S h)$ is also an affine space: the *affine subspace of A induced by f*. It is easy to check that the functions dir, $\|$, $\upuparrows$, and $\upharpoonleft\downharpoonright$ defined relative to F are simply restrictions of the same functions defined relative to A. The front range of F is the part of f contained in the front range of A.

In particular, the affine subspaces of $\mathbf{A}_\nu = (\mathbf{T}_\nu, \Omega)$ have the form $(f, \mathrm{dir}(f))$, where f is a flat not contained in Ω. A two-dimensional affine subspace of $\mathbf{A}_3$, for example, consists of a proper plane π and one of its two lines at infinity, whose orientation must agree with the circular arrow on π as seen from the front range of $\mathbf{T}_\nu$. See figure 1. In what follows, we will implicitly identify a flat f of $\mathbf{T}_\nu$ with the affine subspace of $\mathbf{A}_\nu$ induced by f.

For each affine space $A = (S, h)$ there are three other spaces which differ from A only in the orientation of their parts, namely $(\neg S, h)$, $(S, \neg h)$, and $(\neg S, \neg h)$. Since $\neg S \wedge h = \neg h$, only the last one is a subspace of A (and vice-versa), which I denote by $\neg A$. The other two alternatives, $(\neg S, h)$ and $(S, \neg h)$ are subspaces of each other. Their front and back ranges are switched with respect to A, and their dir function is opposite to that of A.

2.4. Affine maps

Informally, an affine map is a geometric transformation that preserves parallelism. In the two-sided framework, I will define an *affine map* between two affine spaces $A = (S, h)$ and $B = (T, k)$ as a projective map M from S to T that takes h

to k. This is enough to ensure that M takes the front range of A to the front range of B, and that M commutes with the functions dir, $\|$, $\Uparrow$, and $\Updownarrow$, relative to A and B. That is,

$$\underset{B}{\mathrm{dir}}(fM) = (\underset{A}{\mathrm{dir}}(f))M,$$

$$(fM) \Uparrow_B (gM) \Leftrightarrow f \Uparrow_A g,$$

$$(fM) \Updownarrow_B (gM) \Leftrightarrow f \Updownarrow_A g, \text{ and}$$

$$(fM) \|_B (gM) \Leftrightarrow f \|_A g.$$

Let's consider in particular the affine maps of $\mathbf{A}_\nu$ to itself. Recall that a projective map of $\mathbf{T}_\nu$ to $\mathbf{T}_\nu$ is characterized by an $n \times m$ matrix with positive determinant. An affine map M must in addition take the improper points $\mathbf{e}^1, .. \mathbf{e}^\nu$ to infinity; this means M must be of the form

$$M = \begin{bmatrix} m_0^0 & m_1^0 & m_2^0 & \cdots & m_\nu^0 \\ 0 & m_1^1 & m_2^1 & \cdots & m_\nu^1 \\ 0 & m_1^2 & m_2^2 & \cdots & m_\nu^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & m_1^\nu & m_2^\nu & \cdots & m_\nu^\nu \end{bmatrix} \tag{6}$$

Moreover, since Ω is mapped to itself, the front origin $O = \mathbf{e}^0$ must be mapped to a point on the front range. From this and from the fact that M is positive, we conclude that

$$m_0^0 > 0, \tag{7}$$

and

$$\begin{vmatrix} m_1^1 & \cdots & m_\nu^1 \\ \vdots & & \vdots \\ m_1^\nu & \cdots & m_\nu^\nu \end{vmatrix} > 0 \tag{8}$$

Conversely, it is easy to check that any map of the form (6) satisfying (7–8) is an affine map of $\mathbf{A}_\nu$ to itself. For example,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \tag{9}$$

is an affine map of $\mathbf{A}_2$ to $\mathbf{A}_2$ whose effect is depicted in figure 2.

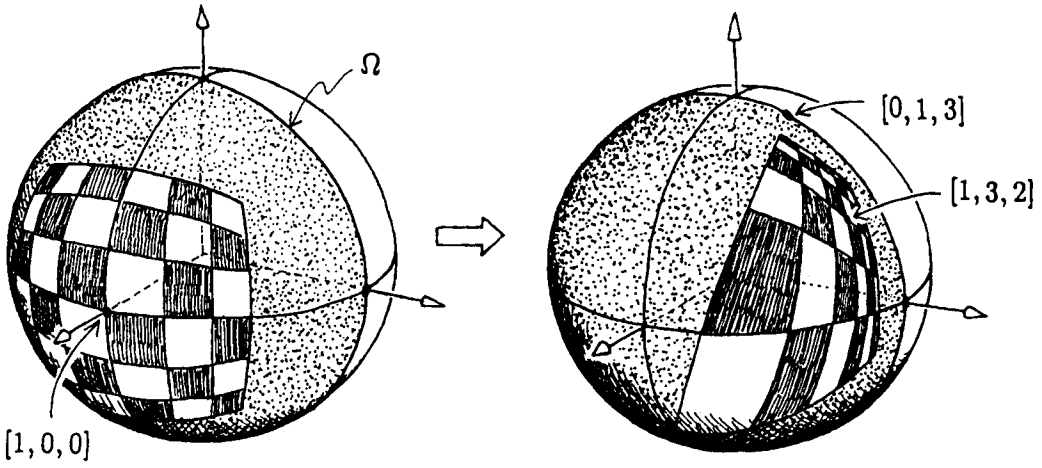

Figure 2. An affine mapping of the plane.

In general, a projective map with the form of (6) takes $\mathbf{A}_\nu$ to one of the four affine spaces $(\mathbf{T}_\nu, \Omega)$, $(\neg\mathbf{T}_\nu, \Omega)$, $(\mathbf{T}_\nu, \neg\Omega)$, $(\neg\mathbf{T}_\nu, \neg\Omega)$. If $m_0^0 > 0$, then the front part of $\mathbf{A}_\nu$ is mapped into itself; this means the image is either $\mathbf{A}_\nu$ or $\neg\mathbf{A}_\nu$, depending on the sign of the cofactor (8). Conversely, if $m_0^0 < 0$, the map exchanges the front and back ranges of $\mathbf{A}_\nu$; this means it takes $\mathbf{A}_\nu$ either to the affine space $(\neg\mathbf{T}_\nu, \Omega)$ or to $(\mathbf{T}_\nu, \neg\Omega)$, depending on the sign of the cofactor (8). For example, consider the map

$$N = \left[\!\!\left[\begin{array}{cccc} -1 & 0 & \cdots & 0 \\ 0 & -1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{array} \right]\!\!\right] \tag{10}$$

This map takes every point of $\mathbf{T}_\nu$ to its antipode, and therefore every flat f of rank r to the flat $\neg^r f$. In particular, N takes $\mathbf{T}_\nu$ to $\neg^n\mathbf{T}_\nu$, and Ω to $\neg^{n-1}\Omega$; that is, it takes the affine space $\mathbf{A}_\nu$ to either $(\mathbf{T}_\nu, \neg\Omega)$ or $(\neg\mathbf{T}_\nu, \Omega)$, depending on whether ν is odd or even.

2.5. Affine frames

An *affine frame* for an affine space $A = (S, h)$ is simply a mixed frame for S with horizon h. Since h is already implicit in the space, an affine frame for A is simply a proper simplex spanning S with no vertices on h.

Let $s = (s^0, .. s^\kappa)$ and $t = (t^0, .. t^\kappa)$ be affine frames for two affine spaces $A = (S, h)$ and $B = (T, g)$. Is there some affine map from A to B that takes s to

t? As we know from the properties of mixed frames, there exists a projective map from S to T that takes s to t and h to g if and only if the two frames have the same signature: that is, if $s^i \diamond_S h = t^i \diamond_T g$ for all i. When this happens, the map exists and is unique.

How do we compute this map? In the most general case, where S and T are κ-dimensional flats of $\mathbf{T}_\mu$ and $\mathbf{T}_\nu$, and the horizons h and g are arbitrary, we have to compute the projective map relating two mixed frames, a problem we already solved in chapter 12. As we discussed in that chapter, it is convenient to break down the problem into two steps, by computing maps M_s and M_t from some "standard" frame f of $\mathbf{T}_\kappa$ to the frames s and t, respectively. The desired map then will be the composition $\overleftarrow{M}_s M_t$. In particular, we may let f be the standard mixed frame with the proper signature, and compute M_s and M_t by formula (12:6).

2.6. Affine maps between subspaces of $\mathbf{A}_\nu$

The formulas of chapter 12 become a little simpler if A and B are affine subspaces of $\mathbf{A}_\nu$ and $\mathbf{A}_\mu$, respectively; that is, if $h = S \wedge \Omega_\nu$ and $g = T \wedge \Omega_\mu$. In that case, the signature σ of the frame s tells whether each point is on the front or back range of $\mathbf{T}_\nu$; that is, $\sigma_i = \text{sign}(s_0^i)$. The map that takes the standard mixed frame mfr_σ of $\mathbf{T}_\kappa$ to s is

$$M_s = \begin{bmatrix} |1/s_0^0| & & 0 \\ & \ddots & \\ 0 & & |1/s_0^\kappa| \end{bmatrix} \begin{bmatrix} s_0^0 & \cdots & s_\nu^0 \\ \vdots & & \vdots \\ s_0^\kappa & \cdots & s_\nu^\kappa \end{bmatrix} \tag{11}$$

The homogeneous coordinates determined by this map on the space A are the *barycentric coordinates* relative to the simplex s, as discussed in section12:4.2.

2.7. Standard affine frames

A minor inconvenience of using mfr_σ as the intermediate frame is that it is not an affine frame of $\mathbf{A}_\kappa$ (recall that the horizon of mfr_κ is $\langle\sigma\rangle$, rather than Ω). Therefore, the map M_s of (11) does not take the affine space $\mathbf{A}_\kappa$ to A, but rather the space $(\mathbf{T}_\kappa, \langle\sigma\rangle)$. If we use (11) only as a tool for computing maps between two affine subspaces A and B, the fact that M_s is not affine is irrelevant, since the composition $\overleftarrow{M}_s M_t$ will be.

However, suppose that for some reason we need a map like M_s all by itself, and we want it to be affine. Then, instead of the standard mixed frame of $\mathbf{T}_\kappa$, we can use the *standard affine frame* of $\mathbf{A}_\kappa$, consisting of the origin

$$\mathbf{u}^0 = [1,0,0,0,0,..] = O$$

and the κ *axial units*

$$\begin{aligned} \mathbf{u}^1 &= [1,1,0,0,0,..] \\ \mathbf{u}^2 &= [1,0,1,0,0,..] \\ \mathbf{u}^3 &= [1,0,0,1,0,..] \\ &\vdots \end{aligned} \tag{12}$$

The points $\mathbf{u}^1,..\,u^\kappa$ are the points at unit distance from the origin on each coordinate axis of the front range. See figure 3.

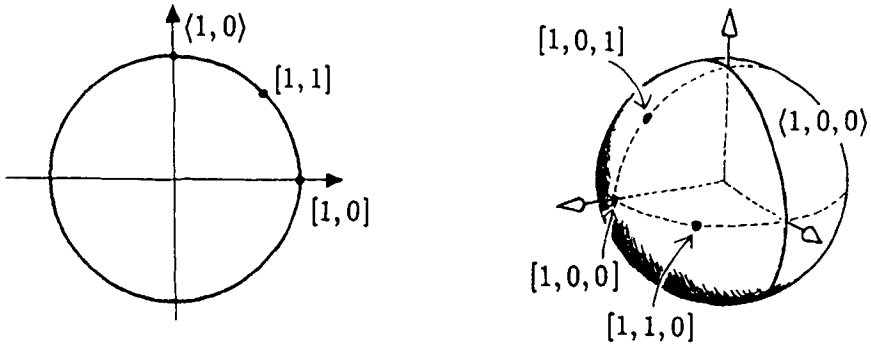

Figure 3. The standard affine frames of $\mathbf{T}_1$ and $\mathbf{T}_2$.

Actually, this is only the first member of a family of 2^n standard affine frames. In general, the standard affine frame with signature σ, denoted by afr_σ, consists of the points $\sigma_0\mathbf{u}^0, \sigma_1\mathbf{u}^1, ..\,\sigma_\nu\mathbf{u}^\nu$.

2.8. The map determined by an affine frame

Suppose $A = (S,h)$ is an affine space, and s an affine frame for A with signature σ. The map from $\mathbf{A}_\kappa$ to A that takes the standard affine frame of the former to the simplex s can be computed by relating both to the standard mixed frame of $\mathbf{T}_\kappa$. That is, we compose the projective map taking afr_σ to mfr_σ with the map taking mfr_σ to (s,h), computed by formula (12:6).

In particular, if A is an affine subspace of $\mathbf{A}_\nu$, the map that takes afr_σ to s turns out to be

$$M_s = \begin{bmatrix} 1/s_0^0 & 0 & 0 & \cdots & 0 \\ -1/s_0^0 & 1/s_0^1 & 0 & \cdots & 0 \\ -1/s_0^0 & 0 & 1/s_0^2 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -1/s_0^0 & 0 & \cdots & & 1/s_0^\kappa \end{bmatrix} \begin{bmatrix} s_0^0 & s_1^0 & \cdots & \cdots & \cdots & s_\nu^0 \\ s_0^1 & s_1^1 & \cdots & \cdots & \cdots & s_\nu^1 \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ s_0^\kappa & s_1^\kappa & \cdots & \cdots & \cdots & s_\nu^\kappa \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{s_1^0}{s_0^0} & \frac{s_2^0}{s_0^0} & \cdots & \frac{s_\nu^0}{s_0^0} \\ 0 & \frac{s_1^1}{s_0^1} - \frac{s_1^0}{s_0^0} & \frac{s_2^1}{s_0^1} - \frac{s_2^0}{s_0^0} & \cdots & \frac{s_\nu^1}{s_0^1} - \frac{s_\nu^0}{s_0^0} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \frac{s_1^\kappa}{s_0^\kappa} - \frac{s_1^0}{s_0^0} & \frac{s_2^\kappa}{s_0^\kappa} - \frac{s_2^0}{s_0^0} & \cdots & \frac{s_\nu^\kappa}{s_0^\kappa} - \frac{s_\nu^0}{s_0^0} \end{bmatrix} \tag{13}$$

The coordinates determined by this map are the *affine coordinates relative to the frame* s. In this coordinate system, point s^i gets coordinates $\sigma_i \circ \mathbf{u}^i$. The weight coordinate of a point will be positive or negative depending on whether the point is on the front or back part of S.

2.9. Affine interpolation

An important application of affine coordinates is the problem of *affine interpolation*: given two points a, b on the front range of $\mathbf{T}_\nu$ divide the segment ab in two parts whose lengths (in the straight model) are in a given ratio $\lambda : 1 - \lambda$. In other words, find the point c on the segment ab that is λ of the way from a to b. See figure 4.

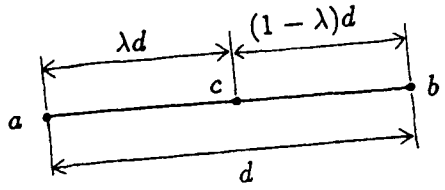

Figure 4. Affine interpolation.

This is a generalization of the midpoint problem we discussed briefly in section 1.2 above. One way to approach this problem is to find an affine map M from $\mathbf{A}_1$ into $\mathbf{A}_\nu$ onto the line $a \vee b$ that takes the standard affine basis of the former to the pair a, b; that is,

$$\begin{aligned} [1,0] &\mapsto a, \\ [1,1] &\mapsto b. \end{aligned} \tag{14}$$

Then the point λ the way from a to b is the image of λ (viewed as a point of the two-sided line $\mathbf{A}_1$) by the map M, that is, $[1, \lambda]M$. According to (13), the map M satisfying (14) is

$$M_{(a;b)} = \begin{bmatrix} 1 & \frac{a_1}{a_0} & \frac{a_2}{a_0} & \cdots & \frac{a_\nu}{a_0} \\ 0 & \frac{b_1}{b_0} - \frac{a_1}{a_0} & \frac{b_2}{b_0} - \frac{a_2}{a_0} & \cdots & \frac{b_\nu}{b_0} - \frac{a_\nu}{a_0} \end{bmatrix}. \tag{15}$$

The point c will then be

$$\begin{aligned} c &= [1,\lambda]M_{(a;b)} \\ &= \left[\, 1\,,\ \frac{a_1}{a_0} + \lambda\Big(\frac{b_1}{b_0} - \frac{a_1}{a_0}\Big)\,,\ \ldots\ \frac{a_\nu}{a_0} + \lambda\Big(\frac{b_\nu}{b_0} - \frac{a_\nu}{a_0}\Big)\,\right] \qquad (16) \\ &= [\, a_0b_0\,,\ \lambda a_0b_1 + (1-\lambda)b_0a_1\,,\ \ldots\ \lambda a_0b_\nu + (1-\lambda)b_0a_\nu\,]\,, \qquad (17) \end{aligned}$$

which is also what we would get by interpolating the Cartesian coordinates in the normal way. Observe that in formula (17) the purpose of multiplying the coordinates of a by b_0, and those of b by a_0, is to normalize both points to have the same weight a_0b_0. After this normalization the homogeneous coordinates from 1 to ν are the same as the Cartesian ones, expressed in a common scale. That being the case, we can obtain the desired point c by interpolating the *homogeneous* coordinates in the given ratio $\lambda : 1-\lambda$.

2.10. Absolute interpolation

Note that formula (17) is multilinear in the coordinates of the given points. It is therefore a generalization of the "natural" midpoint formula (4), and, like it, gives the intuitively correct result only if a and b lie on the front range. (Note that this assumption was necessary to ensure that the frame a, b had positive signature, so that $[1,0]$ and $[1,1]$ got mapped to a and b by M. It was also used in the passage from equation (16) to (17), which is valid only if the product a_0b_0 is positive.) When a and/or b lie on the back range, neither (16) nor (17) give the result one would expect. The first formula always puts c on the front range, while the second puts it on the front if a and b lie on the same range, and on the back if they lie on opposite ranges. That means the midpoint of a and b, for example, may not even lie on the segment ab. This is hardly what one would expect of an "interpolation" formula.

Is there a formula that agrees more closely with the intuitive idea? If a and b are both on the back range, we might just take the antipodal of formula (16) or (17). That corresponds to replacing the map M by $\neg M$, the map from $\mathbf{T}_1$ to the affine line $a \vee b$ with horizon $\neg\,\mathrm{dir}(a \vee b)$ instead of $\mathrm{dir}(a \vee b)$. However, when a and b lie on opposite ranges, we must give up any hope of using an affine map for interpolation, since some point λ of $\mathbf{T}_1$ in the finite range $0 < \lambda < 1$ must be mapped to a point at infinity, and no affine map can do that.

This problem is one we already encountered in chapters 12 and 13, and stems from the way the standard affine frame afr_σ was defined. Recall that the horizon of afr_σ is always Ω, while the vertices of the main simplex may be $\mathbf{u}^i$ or $\neg\mathbf{u}^i$ as necessary to give the desired signature. If we want to have the interval from 0 to

1 mapped to the segment ab (or in general, the interior of the simplex $(u^0;..u^\kappa)$ mapped to the interior of the given simplex s), we must do the opposite: we must use a standard frame whose main simplex stays fixed at $(u^0;..u^\kappa)$, and whose horizon varies according to the desired signature.

It is easy to see that such a frame cannot be an affine frame, except when $\sigma = ++\cdots+$ (when we can use Ω as the horizon) or $\sigma = --\cdots-$ (when we can use $\neg\Omega$). One possible choice that agrees with these two cases is to use $\langle \sigma_0, \sigma_1 - \sigma_0, \sigma_2 - \sigma_0, .. \sigma_\kappa - \sigma_0 \rangle$ as the horizon. The corresponding map is

$$M_{|s|} = \begin{bmatrix}\!\begin{bmatrix} |1/s_0^0| & 0 & 0 & \cdots & 0 \\ -|1/s_0^0| & |1/s_0^1| & 0 & \cdots & 0 \\ -|1/s_0^0| & 0 & |1/s_0^2| & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -|1/s_0^0| & 0 & 0\cdots & & |1/s_0^\kappa| \end{bmatrix}\!\!\begin{bmatrix} s_0^0 & s_1^0 & \cdots & \cdots & \cdots & s_\nu^0 \\ s_0^1 & s_1^1 & \cdots & \cdots & \cdots & s_\nu^1 \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ s_0^\kappa & s_1^\kappa & \cdots & \cdots & \cdots & s_\nu^\kappa \end{bmatrix}\!\end{bmatrix}$$

$$= \begin{bmatrix}\!\begin{bmatrix} \sigma_0 & \frac{s_1^0}{|s_0^0|} & \frac{s_2^0}{|s_0^0|} & \cdots & \frac{s_\nu^0}{|s_0^0|} \\ \sigma_1 - \sigma_0 & \frac{s_1^1}{|s_0^1|} - \frac{s_1^0}{|s_0^0|} & \frac{s_2^1}{|s_0^1|} - \frac{s_2^0}{|s_0^0|} & \cdots & \frac{s_\nu^1}{|s_0^1|} - \frac{s_\nu^0}{|s_0^0|} \\ \vdots & \vdots & \vdots & & \vdots \\ \sigma_\kappa - \sigma_0 & \frac{s_1^\kappa}{|s_0^\kappa|} - \frac{s_1^0}{|s_0^0|} & \frac{s_2^\kappa}{|s_0^\kappa|} - \frac{s_2^0}{|s_0^0|} & \cdots & \frac{s_\nu^\kappa}{|s_0^\kappa|} - \frac{s_\nu^0}{|s_0^0|} \end{bmatrix}\!\end{bmatrix}. \tag{18}$$

In particular, when the simplex s is a pair of points a, b, we get the "absolute" interpolation map

$$M_{|a;b|} = \begin{bmatrix}\!\begin{bmatrix} \operatorname{sign} a_0 & \frac{a_1}{|a_0|} & \cdots & \frac{a_\nu}{|a_0|} \\ \operatorname{sign}(b_0) - \operatorname{sign}(a_0) & \frac{b_1}{|b_0|} - \frac{a_1}{|a_0|} & \cdots & \frac{b_\nu}{|b_0|} - \frac{a_\nu}{|a_0|} \end{bmatrix}\!\end{bmatrix} \tag{19}$$

$$= \begin{bmatrix}\!\begin{bmatrix} a_0\,|b_0| & a_1\,|b_0| & \cdots & a_\nu\,|b_0| \\ b_0\,|a_0| - a_0\,|b_0| & b_1\,|a_0| - a_1\,|b_0| & \cdots & b_\nu\,|a_0| - a_\nu\,|b_0| \end{bmatrix}\!\end{bmatrix}.$$

The point c that is λ the way from a to b, in the "absolute" sense, is then

$$\begin{aligned} c &= [1, \lambda] M_{|a;b|} \\ &= \big[\, a_0\,|b_0| + \lambda\big(b_0\,|a_0| - a_0\,|b_0|\big),\ a_1\,|b_0| + \lambda\big(b_1\,|a_0| - a_1\,|b_0|\big), \end{aligned}$$

$$\cdots\,,\ a_\nu\,|b_0| + \lambda\big(b_\nu\,|a_0| - a_\nu\,|b_0|\big)\ \big] \tag{20}$$

$$= [\ \alpha b_0 + \beta a_0\,,\ \alpha b_1 + \beta a_1\,,\ \alpha b_2 + \beta a_2\,,\ \ldots\,,\ \alpha b_\nu + \beta a_\nu\ ] \tag{21}$$

where $\alpha = \lambda\,|a_0|$, and $\beta = (1-\lambda)\,|b_0|$. That is, we scale each homogeneous tuple by the *positive* factors $|b_0|$ and $|a_0|$, so that their weights have the same *absolute* value but still the original signs, and then we interpolate the homogeneous coordinates in the ratio $\lambda : 1-\lambda$. This agreees with (17) when both points are on the front range, and gives the intuitively correct result when both are on the back range. In either case, as λ goes from 0 to 1 the point c moves from a to b, with uniform "speed."

What happens now when a and b are on opposite ranges? It is not hard to see that as λ goes from 0 to 1 the point c will still traverse the segment ab, but not at a uniform rate: c speeds up as it moves away from a, reaches infinity when $\lambda = \frac{1}{2}$, and then slows down as it moves towards b on the opposite range. Note that this is a generalization of the "absolute" midpoint formula (5). Note also that both (5) and (21) give the null object **0** if a and b are both at infinity, or if a is the antipodal of b rotated 180° around the origin (that is, if $a = [-b_0, b_1, \ldots, b_\nu]$).

2.11. Measure of a simplex

In Cartesian geometry, the area of a triangle with vertices (x^0, y^0) $x^1, y^1)$, and (x^2, y^2) is given by

$$\frac{1}{6}\begin{vmatrix} 1 & x^0 & y^0 \\ 1 & x^1 & y^1 \\ 1 & x^2 & y^2 \end{vmatrix}, \tag{22}$$

with the sign depending on the triangle's orientation. In general, the ν-dimensional measure of a simplex of $\mathbf{R}^\nu$ with vertices $(x_1^i, \ldots x_\nu^i)$ is given by

$$\frac{1}{(\nu+1)!}\begin{vmatrix} 1 & x_1^0 & \cdots & x_\nu^0 \\ 1 & x_1^1 & \cdots & x_\nu^1 \\ \vdots & \vdots & & \vdots \\ 1 & x_1^\nu & \cdots & x_\nu^\nu \end{vmatrix} \tag{23}$$

From this and from the homogeneous-to-Cartesian formulas we get an expression

for the ν-dimensional measure of a simplex s of $\mathbf{A}_\nu$, namely

$$\frac{1}{n!\, s_0^0 s_0^1 \cdots s_0^\nu} \begin{vmatrix} s_0^0 & \cdots & s_\nu^0 \\ \vdots & & \vdots \\ s_0^\nu & \cdots & s_\nu^\nu \end{vmatrix} \tag{24}$$

For consistency, we may want to write this as a point of $\mathbf{T}_1$,

$$\mathrm{vol}(s) = \left[n! \cdot s_0^0 s_0^1 \cdots s_0^\nu \,,\, \begin{vmatrix} s_0^0 & \cdots & s_\nu^0 \\ \vdots & & \vdots \\ s_0^\nu & \cdots & s_\nu^\nu \end{vmatrix} \right] \tag{25}$$

Note that this is a multilinear formula, which means

$$\begin{aligned} \mathrm{vol}(\neg s^0, s^1, s^2, .. s^\nu) &= \mathrm{vol}(s^0, \neg s^1, s^2, .. s^\nu) \\ &= \cdots = \mathrm{vol}(s^0, s^1, s^2, .. \neg s^\nu) = \neg\, \mathrm{vol}(s^0, s^1, s^2, .. s^\nu) \end{aligned} \tag{26}$$

The orientation of s is given by the sign of the second coordinate of $\mathrm{vol}(s)$ only. An alternative, "absolute" formula is

$$\mathrm{vol}(s) = \left[n! \cdot \left| s_0^0 s_0^1 \cdots s_0^\nu \right| \,,\, \begin{vmatrix} s_0^0 & \cdots & s_\nu^0 \\ \vdots & & \vdots \\ s_0^\nu & \cdots & s_\nu^\nu \end{vmatrix} \right] \tag{27}$$

which is always a point of the front range of $\mathbf{T}_1$ (or infinite), and whose numerical sign gives the orientation of s.

Since affine maps preserve the ratio of measures, we can use formula (25) or (27) to compute the ratio between the measures of two simplices s, t contained in the same κ-dimensional affine subspace A of $\mathbf{A}_\nu$. We have only to compute an affine map M that takes A to $\mathbf{A}_\kappa$, compute the measures of $M(s)$ and $M(t)$ in $\mathbf{A}_\kappa$, and take their ratio. We can compute this ratio even if s and t lie in distinct but parallel affine subspaces of $\mathbf{T}_\nu$: it suffices to compute a translation T that takes B to A, and proceed as above with s and $T(t)$.

This is the best we can do within affine geometry. To compute the absolute simplex measure in a proper subspace of $\mathbf{A}_\nu$, or the simplex measure ratio between non-parallel subspaces, we need non-affine concepts such as congruence and distance-preserving maps, which we will study in chapter 17.

Chapter 16
Vector algebra

We can emulate the linear vector space $\mathbf{R}^\nu$ in a ν-dimensional affine space A by selecting a fixed point o on the front range of the latter as the origin. Then we represent any vector v of $\mathbf{R}^\nu$ by a point $\dot{v}$ on the front range of A. The addition of two vectors u, v can be carried out by construcing the parallelogram with corners at o, $\dot{u}$, $\dot{v}$. The product of a vector v by a scalar λ can be done by affine interpolation (or extrapolation) between o and $\dot{v}$ in the ratio $\lambda : 1 - \lambda$. In doing so, we gain the ability to handle infinite-length vectors, which are often handy. We also gain a second copy of $\mathbf{R}^\nu$, namely the points on the back range of A. This feature may be useful in some situations, and is harmless at worst: if we don't need it, we can simply ignore the distinction between v and $\neg v$. Formally,

Definition 1. A *two-sided vector space* is a triple $V = (S, h, o)$ where S is a two-sided space, h (the *horizon*) is a hyperplane of S, and o (the *origin*) is a point of S on the positive side of h.

In particular, we can take $A = \mathbf{A}_\nu$ and $o = O = [1, 0, 0, ..]$, which means we represent a vector $(x_1, x_2, .. x_\nu)$ of $\mathbf{R}^\nu$ by the point $(1, x_1, x_2, .. x_\nu)$ of $\mathbf{T}_\nu$ (this is of course the familiar Cartesian-to-homogeneous mapping). The resulting space is the *canonical two-sided vector space* $\mathbf{V}_\nu = (\mathbf{T}_\nu, \Omega, O)$.

A subspace of a two-sided vector space $V = (S, h, o)$ is any triple (f, g, o) where f is a flat of S that contains o, and $g = f \wedge h$. That is, (f, g) is an affine subspace of (S, h), and o is on the front range of (f, g). In particular, $(\neg S, \neg h, \neg o)$ is a vector subspace of $V = (S, h, o)$, denoted by $\neg V$. Observe that the other variants $(S, h, \neg o)$, $(S, \neg h, o)$, $(S, \neg h, \neg o)$ and so on are not subspaces of V by this definition.

1.1. Translations

In general terms, a translation is a projective map between two co-parallel subspaces of an affine space $A = (S, h)$ that preserves the direction of every subflat. Obviously, a translation is a special case of affine map, one that maps every point of h to itself, and not just the hyperplane h to itself. A translation in a vector space $V = (S, h, o)$ can be uniquely characterized by the image of the origin o. This establishes a correspondence between translations and the points (vectors) on the

front range of V. In particular, the translation of $\mathbf{V}_\nu$ that takes the origin O to the point $x = [x_0, .. x_\nu]$ is

$$\begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_\nu \\ 0 & x_0 & 0 & \cdots & 0 \\ 0 & 0 & x_0 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_0 \end{bmatrix} \tag{1}$$

Conversely, any matrix of this form with $x_0 > 0$ defines a translation. If x_0 is negative (meaning the point x is on the back range) then the corresponding translation (1) is an affine map from $\mathbf{A}_\nu$ to $\neg^n \mathbf{A}_\nu$, that swaps the front and back ranges.

1.2. Vector addition

If we equate the vectors of a two-sided vector space V with its translations, we can add two vectors by composing the corresponding translations. This operation is is given by a simple formula:

$$\begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_\nu \\ 0 & x_0 & 0 & \cdots & 0 \\ 0 & 0 & x_0 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_0 \end{bmatrix} \begin{bmatrix} y_0 & y_1 & y_2 & \cdots & y_\nu \\ 0 & y_0 & 0 & \cdots & 0 \\ 0 & 0 & y_0 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y_0 \end{bmatrix}$$

$$= \begin{bmatrix} x_0y_0 & x_0y_1 + y_0x_1 & x_0y_2 + y_0x_2 & \cdots & x_0y_\nu + y_0x_\nu \\ 0 & x_0y_0 & 0 & \cdots & 0 \\ 0 & 0 & x_0y_0 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_0y_0 \end{bmatrix} \tag{2}$$

The formula for vector addition is therefore

$$\begin{aligned} &[x_0, .. x_\nu] + [y_0, .. y_\nu] \\ &\quad = [\, x_0y_0\,,\ x_0y_1 + y_0x_1\,,\ x_0y_2 + y_0x_2\,,\ \ldots\,,\ x_0y_\nu + y_0x_\nu\,] \end{aligned} \tag{3}$$

The same formula results if we add two vectors by mapping one of them (viewed as a point) by the translation matrix associated with the other.

Formula (3) gives the correct result if x_0 and y_0 are positive, that is, if the two vectors are points of the front range. If only one of x and y is a point at infinity, that point will also be their sum. If both x and y are at infinity, the sum is the undefined point **0**. Formula (3) is multilinear, and hence a "natural" formula in the sense discussed before: we have $(\neg x)+y = x+(\neg y) = \neg(x+y)$ for all x, y. It follows that $x + y$ is a front vector if x and y are on the same range, and is a back vector if they are on different ranges. Note that adding the front origin $O = [1,0,0,..0]$ to a vector leaves it unchanged, whereas adding the back origin $\neg O = [-1,0,0,..0]$ sends the vector to its antipode (the vector with the same Cartesian coordinates, but on the other range).

1.3. Vector negation

The inverse of the translation (1) is given by the matrix

$$\begin{bmatrix} x_0 & -x_1 & -x_2 & \cdots & -x_\nu \\ 0 & x_0 & 0 & \cdots & 0 \\ 0 & 0 & x_0 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_0 \end{bmatrix} \tag{4}$$

This gives a formula for the additive inverse of a vector,

$$-[x_0,..x_\nu] = [x_0, -x_1, -x_2, ..-x_\nu] \tag{5}$$

Indeed, we have

$$\begin{aligned} x+(-x) &= [\, x_0x_0\,,\ x_0x_1 - x_0x_1\,,\ x_0x_2 - x_0x_2\,,\ \ldots\,,\ x_0x_\nu + x_0x_\nu\,] \\ &= [(x_0)^2, 0, 0, \ldots, 0] \end{aligned}$$

which is the front origin O if x is a proper point, and the null object **0** if x is at infinity. The difference of two vectors is then

$$\begin{aligned} &[x_0,..x_\nu] - [y_0,..y_\nu] \\ &\quad = [\, x_0y_0\,,\ y_0x_1 - x_0y_1\,,\ y_0x_2 - x_0y_2\,,\ \ldots\,,\ y_0x_\nu - x_0y_\nu\,] \end{aligned} \tag{6}$$

1.4. "Absolute" vector addition

As one might expect, there is an "absolute" alternative to the vector addition formula (3):

$$\begin{aligned}&[x_0, .. x_\nu] + [y_0, .. y_\nu] \\ &\quad = [\, \tfrac{1}{2}(x_0\,|y_0| + y_0\,|x_0|)\,,\ |x_0|\,y_1 + |y_0|\,x_1\,,\ \ldots\,,\ |x_0|\,y_\nu + |y_0|\,x_\nu\,]\end{aligned} \tag{7}$$

If x and y are both on the front range or both on the back range, formula (7) puts $x + y$ on that same range; otherwise it returns a point at infinity in the direction of $(\neg x) + y$ or $x + (\neg y)$, depending on whether x is on the front or back range, respectively. However, if $x = \neg(-y)$ the result is the null object $\mathbf{0}$.

1.5. Multiplication by a scalar

The *scalar multiplication* of a point $x = [x_0, ..]$ by a real number β is given by $\beta \cdot x = [x_0, \beta x_1, \beta x_2, ..]$. Equivalently, we can use the formula $\beta x = [x_0/\beta, x_1, x_2, ..]$ if $\beta > 0$, or $\beta x = [-x_0/\beta, -x_1, -x_2, ..]$ if $\beta < 0$. Note that $-1 \cdot x = [x_0, -x_1, -x_2, ..]$ is the additive inverse $-x$ of x, as defined above. The proper way to view these formulas is to imagine the scalars as elements of $\mathbf{T}_1$, the two-sided line. Then the scalar multiplication is given by

$$[\beta_0,\ \beta_1] \cdot [x_0,\ x_1,\ x_2,\ ..] = [\beta_0 x_0,\ \beta_1 x_1\,, \beta_1 x_2,\ ..]. \tag{8}$$

The range of $\mathbf{T}_\nu$ on which the product $\beta \cdot x$ lies is determined by the product of the signs of their weights β_0 and x_0. So, for example, multiplication by $[3, -2]$ (which lies on the front range of $\mathbf{T}_1$) produces the vector $(-\frac{2}{3}) \cdot x$ on the same range as x, whereas multiplication by $[-3, 2]$ gives the antipodal of the above vector.

1.6. The two-sided real line

When $\nu = 1$, the operations of addition and multiplication by a scalar defined above become addition and multiplication of two elements of $\mathbf{T}_1$:

$$\begin{aligned}[x_0\,, x_1] + [y_0,\ y_1] &= [x_0 y_0,\ x_0 y_1 + y_0 x_1], \\ [x_0,\ x_1] \cdot [y_0,\ y_1] &= [x_0 y_0,\ x_1 y_1].\end{aligned} \tag{9}$$

As for subtraction and division, it suffices to define

$$\begin{aligned}-[x_0,\ x_1] &= [x_0,\ -x_1], \\ 1/[x_0\,, x_1] &= [x_1,\ x_0].\end{aligned} \tag{10}$$

Note however that

$$x + (-x) \;=\; [x_0\,, x_1] + [x_0,\, -x_1] \;=\; [(x_0)^2,\, 0]$$

which is normally the front origin, but is undefined when $x_0 = 0$. Also,

$$x \cdot (1/x) \;=\; [x_0,\, x_1] \cdot [x_1,\, x_0]) \;=\; [x_0 x_1,\, x_0 x_1]$$

which is $[1,1]$ if x is positive, $[-1,-1] = \neg[1,1]$ if x is negative, and $[0,0] = \mathbf{0}$ if x is infinite or zero ($[0,\pm 1]$ or $[\pm 1, 0]$). With these caveats, formulas (9) and (10) allow us to do arithmetic on $\mathbf{T}_1$ as if it were a two-sided version of the real line.

1.7. Linear maps

The linear maps of $\mathbf{R}^\nu$ are simulated in $\mathbf{T}_\nu$ by projective maps that keep both Ω and the origin fixed. They have the form

$$\left[\!\!\left[\begin{array}{ccccc} a_{00} & 0 & 0 & \cdots & 0 \\ 0 & a_{11} & a_{12} & \cdots & a_{1\nu} \\ 0 & a_{21} & a_{22} & \cdots & a_{2\nu} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{\nu 1} & a_{\nu 2} & \cdots & a_{\nu\nu} \end{array}\right]\!\!\right] \tag{11}$$

where a_{00} is positive, and corresponds to the linear map of $\mathbf{R}^\nu$ to $\mathbf{R}^\nu$ whose coefficient matrix consists of the ratios a_{ij}/a_{00} for $i,j \in \{1..\nu\}$. In particular,

$$\left[\!\!\left[\begin{array}{ccccc} x_0 & 0 & 0 & \cdots & 0 \\ 0 & x_1 & 0 & \cdots & 0 \\ 0 & 0 & x_2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_\nu \end{array}\right]\!\!\right] \tag{12}$$

is the matrix of a *scaling map*, whose effect is to multiply the ith Cartesian coordinate by the ratio x_i/x_0. A convenient way to specify such a map (for example, to a graphics package, or to a procedure that builds scaling maps) is to give the point $[x_0, x_1, .., x_\nu]$, which is the image of the standard unit point $\mathbf{u} = [1, 1, .. 1]$ by the desired map. When $x_1 = x_2 = \cdots = x_\nu$ we get a a *uniform scaling*, whose effect is to scale all vectors by the same ratio x_1/x_0. Again, for consistency we may want to think of this ratio as the two-sided fraction $[x_0, x_1]$.

Chapter 17
Euclidean geometry on the two-sided plane

We can emulate Euclidean geometry in the two-sided space $\mathbf{T}_\nu$, by looking at the two ranges of the straight model as copies of the ν-dimensional Euclidean space. This allows us to define perpendicularity, congruence, angular measure, and other Euclidean concepts by means of the familiar tools of Cartesian geometry. This interpretation turns $\mathbf{T}_\nu$ into a canonical *two-sided Euclidean space.*

If we want to define two-sided Euclidean spaces in a more abstract way, we must find some "fundamental" object from which all other Euclidean concepts can be derived by projective tools. In affine geometry, for example, this role is played by the horizon hyperplane. In Euclidean geometry we can use as fundamental objects a horizon hyperplane, plus a polarity relation restricted to flats on that hyperplane, as shown below.

1. Perpendicularity

First of all, let's define what it means for two flats of $\mathbf{T}_\nu$ to be perpendicular. When $\nu \leq 3$, we want two flats to be perpendicular if and only if their front parts are perpendicular in the Cartesian sense. To do that, we need some auxiliary definitions:

1.1. Orthogonal directions

Let a, b be two directions of $\mathbf{T}_\nu$, that is, two subflats of Ω_ν. By definition, a and b are *orthogonal* if they are polar to each other. In terms of the homogeneous model, this means a and b are orthogonal subspaces of $\mathbf{R}^n$ contained in the subspace $x_0 = 0$. In particular, two points x, y of Ω are orthogonal if and only if their homogeneous coordinates satisfy $x_1 y_1 + \cdots + x_\nu y_\nu = 0$.

1.2. Normals

Let a be a proper flat of $\mathbf{T}_\nu$. Its *normal direction* (or simply its *normal*) is the improper flat norm(a) satisfying

$$\begin{aligned} \mathrm{dir}(a) &\perp \mathrm{norm}(a) \\ \mathrm{dir}(a) \vee \mathrm{norm}(a) &= \Omega. \end{aligned} \tag{1}$$

In other words, norm(a) is $\Omega \lceil \mathrm{dir}(a)$, the right polar complement of a relative to Ω. This is the flat set consisting of all points at infinity that are orthogonal to dir(a). It follows from this definition that

$$\begin{aligned} \mathrm{norm}(a) &= (O \vee \mathrm{dir}(a))^{\vdash} \\ a \vee \mathrm{norm}(a) &= \mathbf{T}_\nu. \end{aligned} \tag{2}$$

This definition of "normal direction" generally agrees with the usual one, as applied to the straight model, except for our convention of using points at infinity to represent directions. For example, the normal of a line l of $\mathbf{T}_2$ is a point at infinity in the direction 90° counterclockwise from l, as seen from the front range. See figure 1.

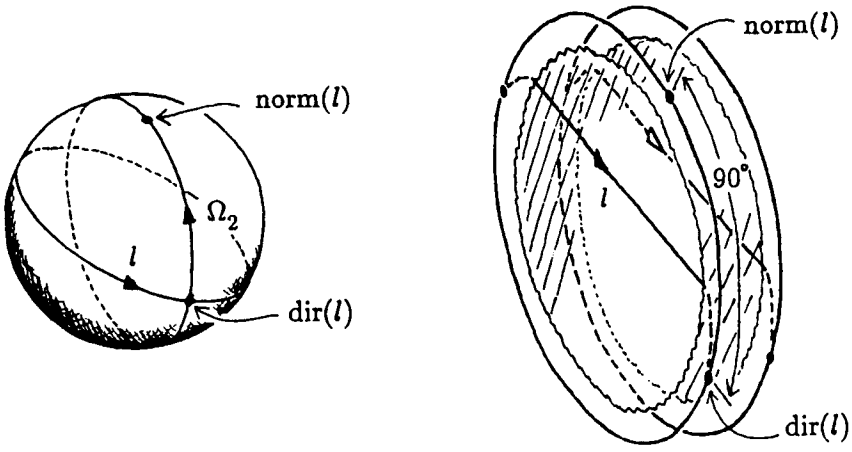

Figure 1. The normal of a line.

The normal norm(π) of a proper plane π of $\mathbf{T}_3$ is a point at infinity whose direction is perpendicular to π (in the straight model of $\mathbf{T}_3$). The orientation of norm(π) is derived from that of π by the right-hand rule, as shown in figure 2. The normal of a line l in $\mathbf{T}_3$ is a line at infinity, which is the direction of any proper plane that is perpendicular to l, in the usual sense. The line l and its normal are positively oriented in $\mathbf{T}_3$. See figure 3.

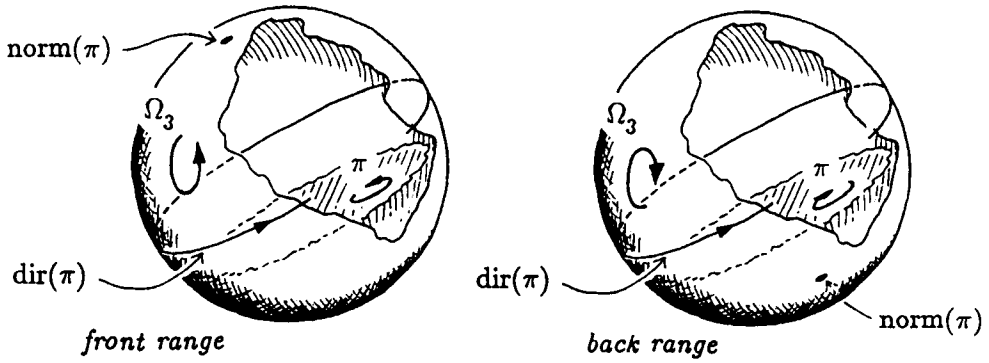

Figure 2. Normal of a plane in $\mathbf{T}_3$.

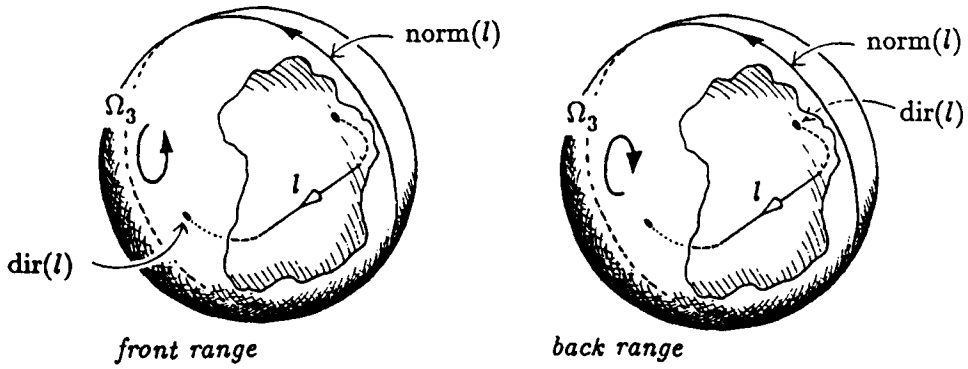

Figure 3. The normal of a line in $\mathbf{T}_3$.

Observe that $\text{norm}(\pi)$ is on the *negative* side of π, but $\text{norm}(l)$ is on the positive side of l. In general,

$$a \diamond \text{norm}(a) = +1$$

$$\text{norm}(a) \diamond a = (-1)^{\text{rank}(a)\,\text{corank}(a)}.$$

In particular, $\text{norm}(h) \diamond h = (-1)^{\nu}$ for any hyperplane h of $\mathbf{T}_\nu$.

1.3. Perpendicular flats

In general, I will say that two proper flats a, b are *perpendicular* if $\text{dir}(a) \subseteq \text{norm}(b)$, or $\text{norm}(a) \subseteq \text{dir}(b)$. For example, two proper lines l, m of $\mathbf{T}_\nu$ are perpendicular if and only if their directions are two polar points of Ω. In the straight model, this condition means there is a plane that contains one line and is perpendicular to the other. Note that two perpendicular lines of $\mathbf{T}_3$ may or may not intersect. See figure 4.

A line is perpendicular to a plane of $\mathbf{T}_3$ if and only if the normal direction of the plane is the direction of the line, or its opposite. In the straight model, this

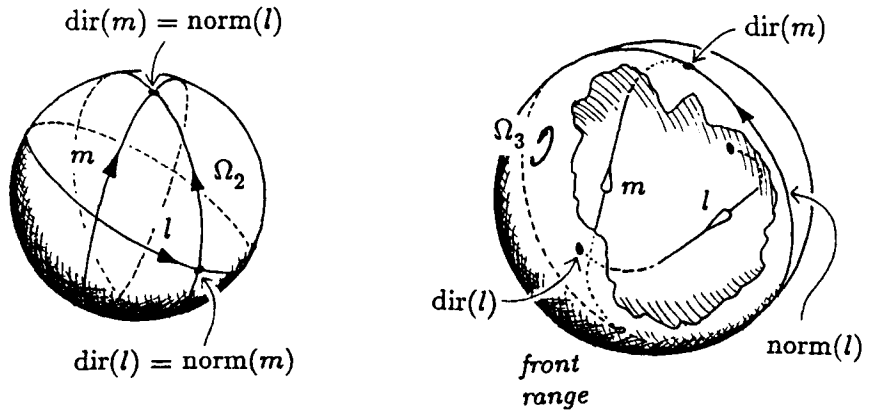

Figure 4. Perpendicular lines of $\mathbf{T}_2$ and $\mathbf{T}_3$.

agrees with the usual definition. See figure 5.

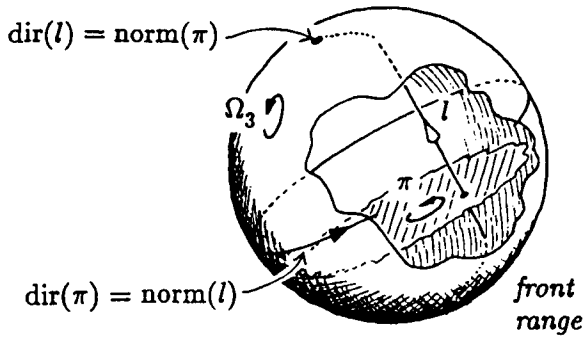

Figure 5. Perpendicularity between a line and a plane of $\mathbf{T}_3$.

Two planes of $\mathbf{T}_3$ are perpendicular if and only if their normal directions are polar (orthogonal) to each other. This is equivalent to saying that the directions of the two planes (two lines at infinity) meet at a right angle on the sphere at infinity Ω. See figure 6.

Since normals and directions are complementary with respect to Ω, we have $\text{dir}(a) \subseteq \text{norm}(b) \Leftrightarrow \text{dir}(b) \subseteq \text{norm}(a)$. It follows that perpendicularity, like parallelism, is a symmetric relation: a is perpendicular to b if and only if b is perpendicular to a.

Note that if a is perpendicular to b, and b to c, we cannot in general conclude that a is parallel to c. For a counterexample, let a, b, and c be the Cartesian axes of $\mathbf{T}_3$. However, the conclusion $a \parallel b$ *is* legitimate for some rank combinations, namely when $\text{rank}(a) \geq 1 + \text{corank}(b) \geq \text{rank}(c)$, or $\text{rank}(a) \leq 1 + \text{corank}(b) \leq \text{rank}(c)$.

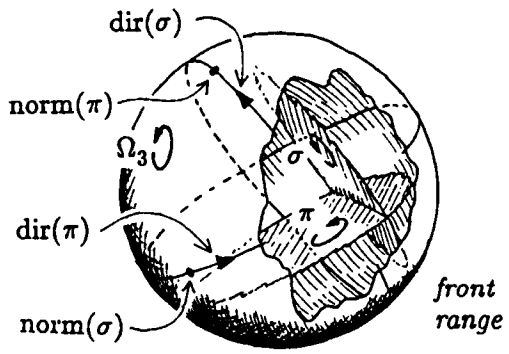

Figure 6. Perpendicular planes of $\mathbf{T}_3$.

Two common instances are when a, b, c are lines of $\mathbf{T}_2$, or a and c are lines and b is a plane of $\mathbf{T}_3$.

Formulas for many familiar constructions of Euclidean geometry follow readily from the definitions above. For example, the flat of maximum rank that is perpendicular to a proper flat a and passes through a proper point p is given by the formula $p \vee \text{norm}(a)$. In particular, the perpendicular bisector of a segment pq is given by $m \vee \text{norm}(p \vee q))$, where m is the segment's midpoint. The perpendicular projection onto a flat a is the map

$$x \mapsto (x \vee \text{norm}(a)) \wedge a.$$

And so forth.

2. Two-sided Euclidean spaces

All concepts of Euclidean geometry can be derived from the notion of perpendicularity. As we saw, perpendicularity in turn can be derived from the polarity predicate $\perp$, restricted to the points of Ω. We can generalize these definitions to arbitrary two-sided spaces by letting any fixed hyperplane h play the role of Ω, and any suitable relation η on flats of h play the role of the polarity predicate. Formally,

Definition 1. The *canonical two-sided Euclidean space* of dimension ν is the triple $\mathbf{E}_\nu = (\mathbf{T}_\nu, \Omega, \perp_\Omega)$, where $\perp_\Omega$ is the standard polarity relation restricted to subflats of Ω.

Definition 2. A *two-sided Euclidean space* is a triple (S, h, ρ) isomorphic to the canonical space $\mathbf{E}_\nu$, for some ν.

Here "isomorphic" means there is an isomorphism η from the affine space (S, h) to

$\mathbf{A}_\nu = (\mathbf{T}_\nu, \Omega_\nu)$, such that $x \rho y \Leftrightarrow (x\eta) \perp (y\eta)$, for all points x, y of h. Note that a two-sided Euclidean space is a two-sided affine space with the extra structure given by the orthogonality predicate ρ.

Definition 3. A *subspace* of an Euclidean space (S, h, ρ) is a triple (T, g, σ) where T is a projective subspace of S, g is $T \wedge_S h$, and σ is the restriction of the relation ρ to the subflats of g.

As in the affine case, there are three other spaces with the same point set as $\mathbf{E}_\nu$ and the same points at infinity, namely $(\pm\mathbf{T}_\nu, \pm\Omega, \perp_\Omega)$. Of these only $(\neg\mathbf{T}_\nu, \neg\Omega, \perp_\Omega)$ is a subset of $\mathbf{E}_\nu$, denoted by $\neg\mathbf{E}_\nu$.

3. Euclidean maps

In classical geometry a Euclidean map can be defined as a map that preserves congruence: two segments have equal length if and only if their images have equal length. Examples are translations, rigid rotations, and uniform scalings. Euclidean maps also preserve perpendicularity, and indeed they are the only maps that do so. Therefore, it is reasonable to use this property to define Euclidean maps of $\mathbf{T}_\nu$:

Definition 4. A *Euclidean map* or *similarity* between two Euclidean spaces $E = (S, h, \rho)$ and $F = (T, g, \sigma)$ is an isomorphism φ from S to T that changes the polarity relation ρ into σ.

That is, $S\varphi = T$, $h\varphi = g$, and $(x\varphi)\ \sigma\ (y\varphi) \Leftrightarrow x\ \rho\ y$ for all $x, y \subseteq h$. In words, a Euclidean map is a projective map that takes orthogonal directions to orthogonal directions. Two obvious examples of Euclidean maps are the translations and uniform scalings defined in chapter 15. Those maps keep all points of Ω fixed, and therefore trivially preserve the polarity relation among those points.

Note that every Euclidean map is affine, but not vice-versa. For example,

$$\begin{bmatrix}\!\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\!\end{bmatrix} \tag{3}$$

is a map that takes the orthogonal directions $[0,1,0]$ and $[0,0,1]$ to the directions $[0,1,1]$ and $[0,0,1]$, which are not orthogonal.

3.1. Analytic characterization

Analytically, a Euclidean map of $\mathbf{T}_\nu$ has a matrix of the form

$$M = \begin{bmatrix} m_0^0 & m_1^0 & m_2^0 & \cdots & m_\nu^0 \\ 0 & m_1^1 & m_2^1 & \cdots & m_\nu^1 \\ 0 & m_1^2 & m_2^2 & \cdots & m_\nu^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & m_1^\nu & m_2^\nu & \cdots & m_\nu^\nu \end{bmatrix} \tag{4}$$

where m_0^0 and its cofactor are positive, and rows 1 through ν are orthogonal vectors with identical length λ. That is,

$$\sum_{1\le k\le \nu} m_k^i m_k^j = 0 \quad \text{and} \quad \sum_{1\le k\le \nu} (m_k^i)^2 = \lambda^2 \qquad \text{for } 1 \le i < j \le \nu. \tag{5}$$

It is easy to check that any map of this form takes Υ_ν to Υ_ν, Ω to Ω, and preserves polarities on Ω. Conversely, it is easy to show that any map that preserves the polarities between all points $[\mathbf{e}^i]$ and $[\mathbf{e}^i \pm \mathbf{e}^j]$, for $1 \le i < j \le n$, must be of this form.

Now consider a map M that satisfies (4) and (5) but has a negative coefficient m_0^0 or a negative determinant. Such a map still takes improper points to improper points and preserves their polarities, but may reverse the orientation of Υ_ν or Ω. That means M is a map from $\mathbf{E}_\nu$ to one of the four Euclidean subspaces $(\pm\mathbf{T}_\nu, \pm\Omega, \perp_\Omega)$. For example, consider the vector negation map N which sends every vector of $\mathbf{V}_\nu$ to its additive inverse, and the antipode map A, which sends every point to its antipode:

$$N = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix} \qquad A = \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix}. \tag{6}$$

Note that

$$\begin{aligned} \Upsilon_\nu N &= \neg^\nu \Upsilon_\nu & \Upsilon_\nu A &= \neg^n \Upsilon_\nu & \Upsilon_\nu AN &= \neg \Upsilon_\nu \\ \Omega_\nu N &= \neg^\nu \Omega_\nu & \Omega_\nu A &= \neg^\nu \Omega_\nu & \Omega_\nu AN &= \Omega_\nu. \end{aligned} \tag{7}$$

So, N is a map from $\mathbf{E}_\nu$ to $\neg^\nu \mathbf{E}_\nu$, and A is a map from $\mathbf{E}_\nu$ to $\neg^\nu(\neg\mathbf{T}_\nu, \Omega, \perp_\Omega)$.

In what follows, a "Euclidean map of a space $E = (S, h, \rho)$" means a Euclidean map from a space E to any of its four related spaces $(\pm S, \pm h, \rho)$.

3.2. Isometries

An *isometry* is a Euclidean map that preserves volumes. Analytically, this condition means that the quantity λ^2 in formula (5) is $(m_0^0)^2$. Translations are obviously isometries, and so are the maps N and A defined above. It is easy to show that every Euclidean map is the composition of an isometry with a uniform scaling. More precisely, a map of $\mathbf{E}_\nu$ that enlarges the volume of every simplex by a factor of τ is the product of an isometry of $\mathbf{E}_\nu$ and a uniform scaling by $|\tau|^{1/n}$.

3.3. Rotations

We may define a *rotation* as an isometry of a space to itself, that is, an isometry that preserves the orientations of the universe and of the horizon. An example is any orthonormal transformation of $\mathbf{R}^\nu$ with determinant $+1$, applied to both ranges of the straight model. However, rotations also include the translations of $\mathbf{A}_\nu$, various helical (screw-like) motions, and also some maps that swap the front and back ranges, such as the antipodal map A above (when ν is even), or the map

$$\begin{bmatrix} -1 & & & & 0 \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}. \tag{8}$$

For any pair of distinct axes i, j in $\{1, \nu\}$, and any angle θ, the map

$$\begin{array}{c} \\ \\ i \to \\ \\ j \to \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & i\downarrow & & j\downarrow & & \\ \end{array} \\ \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \cos\theta & & -\sin\theta & & \\ & & & & & & \\ & & -\sin\theta & & \cos\theta & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{bmatrix} \end{array} \tag{9}$$

rotates each range of the straight model by the angle θ around a $(\nu-2)$-dimensional "axis," the flat $(\mathbf{e}^i \vee \mathbf{e}^j)^\vdash$.

Note that rotations are closed under composition and inverse. In fact, every rotation can be written as the product of a translation, a rotation that doesn't move the origin, and zero or one instances of the rotation (8) above (or any other fixed rotation that swaps the front and back ranges).

3.4. Reflection across a hyperplane

Isometries that map the universe S to $\neg S$ are called *reflections*. An important example is reflection across a proper hyperplane $h = \langle h \rangle = \langle h^0, .. h^\nu \rangle$ of $\mathbf{E}_\nu$. Let g be the hyperplane co-parallel to h that passes through O, that is, $g = \langle g \rangle$ where $\boldsymbol{g} = (0, h^1, .. h^\nu)$. Then the (*Euclidean*) *reflection across* h is by definition the map

$$R_h = [\![(\boldsymbol{g}\boldsymbol{h}^{\mathrm{tr}})I - 2\boldsymbol{h}^{\mathrm{tr}}\boldsymbol{g}]\!]. \tag{10}$$

Expanded, this formula becomes

$$R_h = \begin{bmatrix}\!\begin{bmatrix} \lambda^2 & -2h^0h^1 & -2h^0h^2 & \cdots & -2h^0h^\nu \\ 0 & \lambda^2 - 2h^1h^1 & -2h^1h^2 & \cdots & -2h^1h^\nu \\ \vdots & -2h^2h^1 & \lambda^2 - 2h^2h^2 & & \vdots \\ \vdots & \vdots & & \ddots & \\ 0 & -2h^\nu h^1 & \cdots & & \lambda^2 - 2h^\nu h^\nu \end{bmatrix}\!\end{bmatrix} \tag{11}$$

where $\lambda^2 = \boldsymbol{g}\boldsymbol{h}^{\mathrm{tr}} = \sum_{k=1}^{\nu} h^k h^k$. It is straightforward to check that R_h is an isometry, that it keeps every point on h fixed, and that it swaps the point $\mathrm{norm}(h) = [-g] = [0, -h^1, .. -h^\nu]$ with its antipode. See figure 7.

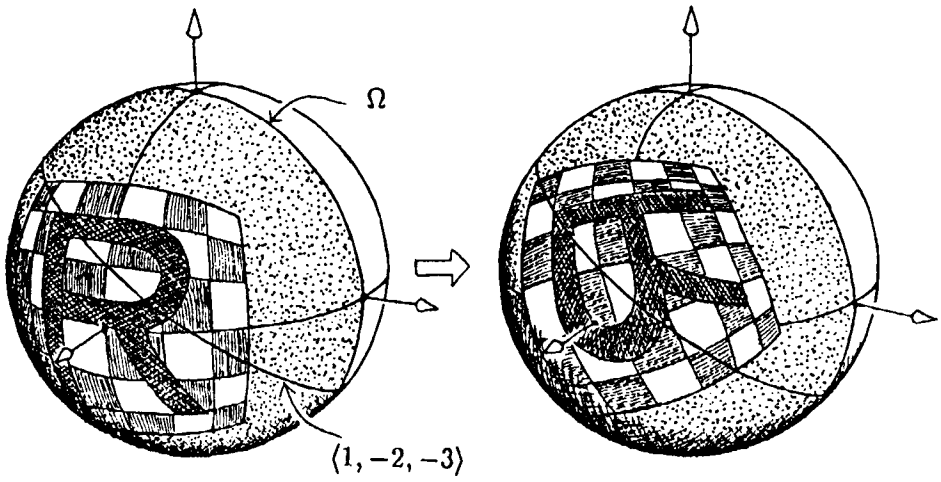

Figure 7. Reflection of $\mathbf{T}_2$ across the line $\langle 1, -2, -3 \rangle$.

Note that all coefficients of R_h in formula (11) are homogeneous second-degree polynomials in the coefficients of h. This means the orientation of h is irrelevant, i.e. $R_h = R_{\neg h}$.

3.5. Reflection across Ω

Another example of reflection is the map

$$R_\Omega = AN = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \tag{12}$$

which in the straight model sends the point $(x_1, .. x_\nu)$ of the front range to point $(-x_1, .. -x_\nu)$ of the back, and vice versa. That is, it sends vector x of the two-sided vector space $\mathbf{V}_\nu$ to the antipodal of its additive inverse, the vector $\neg(-x)$.

In the spherical model, this map simply mirrors the unit sphere across the plane $x_0 = 0$ of $\mathbf{R}^n$. We may call this map *reflection across* Ω. Note that it takes $\mathbf{E}_\nu$ to the space $(\neg\mathbf{T}_\nu, \Omega, \perp_\Omega)$. See figure 8.

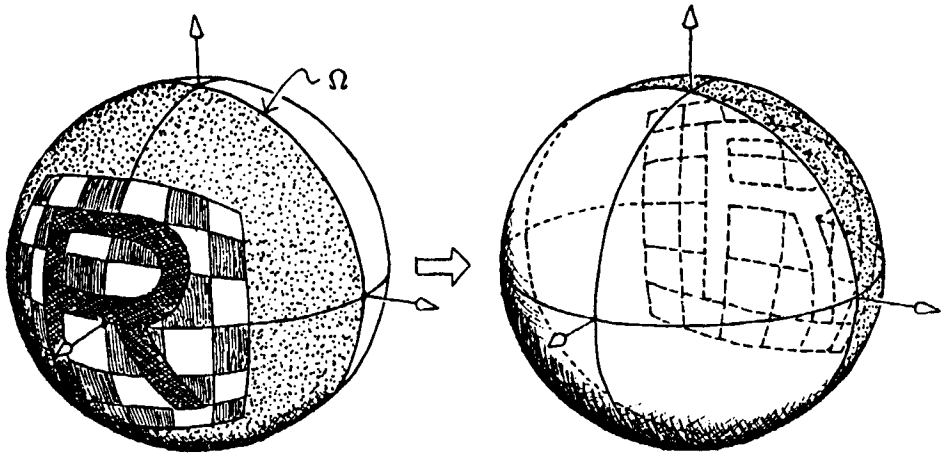

Figure 8. Reflection of $\mathbf{T}_2$ across Ω.

The product of two reflections is a rotation. For example, the product $R_h R_g$, where h and g are parallel hyperplanes, is equivalent to a translation by twice the displacement from h to g. If h and g intersect in a proper hyperline l, the product $R_h R_g$ is (in the straight model) a rotation of each range around the "axis" l, by twice the angle between h and g. In particular, reflection across two perpendicular

hyperplanes gives a rotation of 180° around their common hyperline l: every point x is mapped to the point y such that l is perpendicular to the segment xy, and passes through its midpoint.

This holds in general for the result of reflecting $\mathbf{T}_\nu$ across k pairwise perpendicular hyperplanes, with l replaced by the flat set of co-rank k that is their common intersection. It is customary to call the resulting map R_l the *reflection across* l, even when it is a rotation (k even). For example, by reflecting across all the proper main hyperplanes $\langle \mathbf{e}^1 \rangle$, $\langle \mathbf{e}^2 \rangle$, ..., $\langle \mathbf{e}^\nu \rangle$, we get the *reflection across the origin*,

$$R_O = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix} \tag{13}$$

It can be shown that any isometry of $\mathbf{E}_\nu$ is the product of at most $\nu + 1$ reflections across proper hyperplanes, and at most one instance of R_Ω. The isometry is a rotation or a reflection depending on whether the total number of terms is even or odd, and it swaps the back and front ranges if and only if it includes R_Ω.

4. Length and distance

Although we defined Euclideans maps as transformations that preserve perpendicularity, they turn out to preserve distances as well. To make this statement meaningful, we must define "distance" in a two-sided Euclidean space. It turns out that there are at least two distinct definitions, each with its own advantages and disadvantages. This is another instance of the "natural versus absolute" dilemma we faced in the definition of frames and cross ratio.

4.1. Distance through isometries

Let's consider the two-sided line Υ_1 embedded in $\mathbf{T}_\nu$ in the standard way, that is, as the line $O \vee \mathbf{e}^1$. For any vector x of $\mathbf{V}_\nu$ except $\pm O$, there is exactly one unoriented hyperplane h of $\mathbf{T}_\nu$ such that the reflection of x across h lies on the closed half-line of Υ_1 that goes from $[1,0]$ through $[1,1]$ to $[-1,0]$. In the straight model, that hyperplane is the bisector of the angle determined by the line Υ_1 and

the line $O \vee x$. See figure 9.

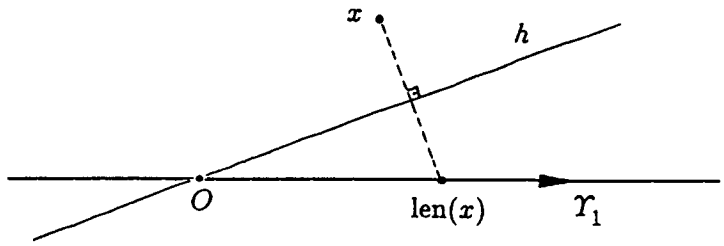

Figure 9. Reflecting a point of Υ_ν onto Υ_1.

The matrix of the reflection is

$$\begin{bmatrix} 1 & 0 \\ 0 & x_1/\lambda \\ 0 & x_2/\lambda \\ \vdots & \vdots \\ 0 & x_\nu/\lambda \end{bmatrix} \tag{14}$$

where $\lambda = \sqrt{x_1 x_1 + \cdots x_\nu x_\nu}$. By definition, the *length of the vector* x is the image of x under this map, namely

$$\text{len}(x) = [\, x_0, \sqrt{x_1 x_1 + \cdots x_\nu x_\nu}\,] \tag{15}$$

Note that the length is sensitive to the sign of the weight coordinate, and no other. Specifically, the length of a front vector is a positive front number, and that of a back vector is a negative back number:

$$\begin{aligned} \text{len}\,[1,1,1] &= [1,\sqrt{2}] &&= \sqrt{2} \\ \text{len}\,[1,-1,-1] &= [1,\sqrt{2}] &&= \sqrt{2} \\ \text{len}\,[-1,1,1] &= [-1,\sqrt{2}] &&= \neg(-\sqrt{2}) \\ \text{len}\,[-1,-1,-1] &= [-1,\sqrt{2}] &&= \neg(-\sqrt{2}). \end{aligned}$$

4.2. Inner product of vectors

Another equivalent way of defining length in $\mathbf{E}_2$ is through the inner product in two-sided vector spaces. From the Cartesian coordinate formulas, the *inner* or *dot product* of two front vectors x and y of $\mathbf{V}_\nu$ is given by

$$x \bullet y = \frac{x_1y_1 + x_2y_2 + \cdots + x_\nu y_\nu}{x_0y_0}. \tag{16}$$

This can be written as the two-sided fraction

$$[\,x_0y_0,\ x_1y_1 + x_2y_2 + \cdots + x_\nu y_\nu\,]. \tag{17}$$

Note that $x \bullet y$ is a front number if both vectors are on the same range, and a back number if they lie on different ranges. In either case, the numerical value (disregarding range) is the classical dot product of the two vectors (ignoring their ranges). Said another way, sending x or y to its antipode also sends $x \bullet y$ to its antipode, but doesn't change its numerical value. Note that the dot product is infinite if any operand is infinite, except that the dot product of two orthogonal infinite vectors is the null object $\mathbf{0} = [0, 0]$.

Note that the dot product of x with itself is precisely the square of the length of x, as defined by formula (15).

4.3. Euclidean distance

The Euclidean distance between two points $[x_0, .. x_\nu]$ and $[y_0, .. y_\nu]$ of the front range is the length of their difference as vectors. From (15) and from the formula for vector difference (16:6), this boils down to

$$\begin{aligned} \operatorname{dist}(x,y) &= \operatorname{len}(x - y) \\ &= \operatorname{len}[\,x_0y_0,\ y_0x_1 - x_0y_1,\ y_0x_2 - x_0y_2,\ \ldots,\ y_0x_\nu - x_0y_\nu\,] \\ &= [\,x_0y_0,\ \sqrt{(y_0x_1 - x_0y_1)^2 + \cdots + (y_0x_\nu - x_0y_\nu)^2}\,]. \end{aligned} \tag{18}$$

Note that

$$\operatorname{dist}(x,y) \;=\; \operatorname{dist}(y,x)$$

and

$$\operatorname{dist}(x,\neg y) \;=\; \operatorname{dist}(\neg x,y) \;=\; \neg(-\operatorname{dist}(x,y))$$

for all x, y. Also, $\operatorname{dist}(x,x) = 0$, $\operatorname{dist}(x,\neg x) = \neg 0$ for any proper point x. From the properties of vector sum in $\mathbf{V}_2$ it follows that the distance as defined above is a front positive number if x and y are on the same range, a back negative number if they are in opposite ranges, plus infinity if one of them is at infinity, and undefined

if both are at infinity. More precisely, the distance increases monotonically from 0 to ∞ as y moves away from x on the same range, and then increases monotonically from $\neg\infty$ to $\neg 0$ as y moves on the opposite range from Ω towards $\neg x$. See figure 10.

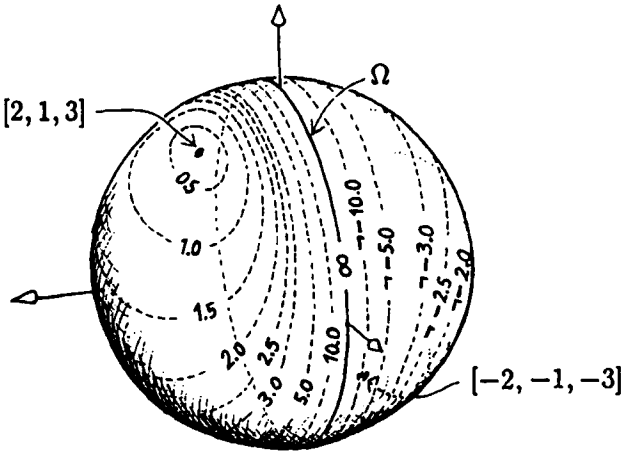

Figure 10. Distances from [2, 1, 3] on the two-sided plane.

Note that the curves of equal distance in figure 10 project as concentric circles in the straight model.

4.4. Closeness and shortness

This jump from $+\infty$ to $-\infty$ as points move to opposite ranges may be an inconvenience in some geometric algorithms, since it means the numerical distance does not increase monotonically as the two points move apart. In such cases, we can obtain a monotonic measure of separation by taking the numerical inverse of formula (18), which we may call the *closeness* of the two points:

$$\begin{aligned} \mathrm{cls}(x, y) &= 1/\operatorname{dist}(x, y) \\ &= [\ \sqrt{(y_0x_1 - x_0y_1)^2 + \cdots + (y_0x_\nu - x_0y_\nu)^2}\ ,\ x_0y_0\ ]. \end{aligned} \tag{19}$$

This quantity is always a front or infinite number. It decreases monotonically from $+\infty$ to $-\infty$ as y moves straight away from x towards $\neg x$ (or vice-versa). It is positive if x and y are on the same range, zero if one of them is at infinity, negative if they

are on opposite ranges, and undefined if both are at infinity. See figure 11.

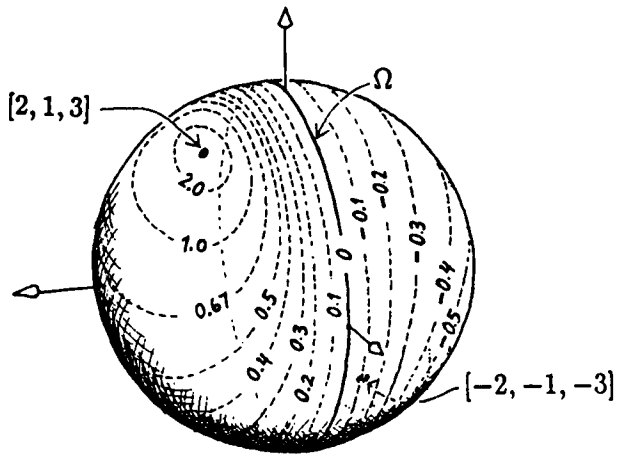

Figure 11. Closeness to [2, 1, 3] on the two-sided plane.

Similarly, we can define the *shortness* of a vector x of $\mathbf{V}_\nu$ as the inverse of its length, that is, the closeness of x and the front origin:

$$\mathrm{shr}(x) = 1/\mathrm{len}(x) = \left[\sqrt{x_1 x_1 + \cdots + x_\nu x_\nu}\,,\ x_0\right]. \tag{20}$$

The shortness decreases monotonically from $+\infty$ to $-\infty$ as x moves straight away from O towards $\neg O$. It is positive for front vectors, zero for infinite vectors, and negative for back vectors.

4.5. Congruence and length ratio

Strictly speaking, length and distance are *not* proper concepts of Euclidean geometry. They depend on a distinguished unit of length, and are not conserved by arbitrary Euclidean maps. Indeed, our definitions of length and distance are based on some concepts that cannot be derived from $\mathbf{E}_\nu$ itself, such as the canonical embedding of Υ_1, the homogeneous coordinates of a point, and so on. As a consequence, we cannot extend those definitions to general Euclidean spaces. What we *can* do in abstract Euclidean geometry is measure the *ratio* of two distances, and in particular check if they are congruent. This ratio is given by the formula

$$\left[a_0 b_0 \sqrt{\textstyle\sum_{k=1}^{\nu}(x_0 y_i - y_0 x_i)^2}\,,\ x_0 y_0 \sqrt{\textstyle\sum_{k=1}^{\nu}(a_0 b_i - b_0 a_i)^2}\right].$$

This ratio is conserved by arbitrary Euclidean maps. It is a front positive point of $\mathbf{T}_1$ if and only if a lies on the same range as b and x lies on the same range as y.

5. Angular measure and congruence

In classical Euclidean geometry we can measure not only lengths and distances, but angles as well. We can compare the angles between two pairs of lines by superimposing one on the other by means of a Euclidean transformation. In the same way we can compute the ratio between two angles, and the measure of an angle.

Extending these notions to two-sided geometry is relatively straightforward. In fact, we will see that in two-sided geometry angles can be handled somewhat more elegantly than in classical geometry, because they can be treated as points of the space, and hence operated upon with the geometric tools we already have at our disposal.

5.1. Angles as points at infinity

Observe that the angle between two proper lines is only a function of their directions, and not of their absolute positions. Therefore, measuring the angle between two lines of $\mathbf{E}_\nu$ corresponds to measuring the separation between two points at infinity. That is, the angular metric for lines of $\mathbf{E}_\nu$ is equivalent to a distance metric on Ω.

Let $x = [0, x_1, x_2]$ be a point at infinity of $\mathbf{T}_2$. The angle between the cardinal direction $\mathbf{e}^1 = [0, 1, 0]$ and x is

$$\arg(x) = \begin{cases} \text{arc tan}(x_2/x_1) & \text{if } x_1 > 0, \\ \text{sign}(x_2) \cdot (\pi/2) & \text{if } x_1 = 0, \text{ and} \\ \text{arc tan}(x_2/x_1) + \pi & \text{if } x_1 < 0. \end{cases} \tag{21}$$

This function is available in most programming languages as a two-argument arctangent procedure.

Of course, the angle $\arg(x)$ is determined only up to a multiple of 2π radians. Measuring angles by (21), or any real-valued formula, has the inconvenience that it introduces two spurious pieces of information: the unit of angular measure (radians, degrees, cycles, etc.), and the disambiguating criterion that picks one value out of the infinitely many equivalent ones. For many uses, it is both simpler and cleaner to represent angles as points of Ω_2, without reducing them to real numbers. That is, we let the point $[0, x_1, x_2]$ represent the angle between the vectors $(1, 0)$ and (x_1, x_2) of $\mathbf{R}^2$. In other words, we represent the angle α by the point $[0, \cos\alpha, \sin\alpha]$.

5.2. Angle arithmetic

With this representation, it is still possible to add, subtract and compare angles using only the four arithmetic operations. Let us denote the operations on angles by $\alpha \hat{+} \beta$, $\alpha \hat{-} \beta$ and $\hat{-}\alpha$; their corresponding formulas are

$$\begin{aligned} [0, x_1, x_2] \hat{+} [0, y_1, y_2] &= [0, x_1y_1 - x_2y_2, x_2y_1 + x_1y_2] \\ [0, x_1, x_2] \hat{-} [0, y_1, y_2] &= [0, x_1y_1 + x_2y_2, x_2y_1 - x_1y_2] \\ \hat{-}[0, x_1, x_2] &= [0, x_1, -x_2]. \end{aligned} \tag{22}$$

To understand these formulas we have only to reinterpret the angle $[0, x_1, x_2]$ as the argument of the complex number $x_1 + \mathbf{i}x_2$. Then equations (22) are simply the formulas for multiplication, division, and conjugation of complex numbers, except that the formula for $x \hat{-} y$ omits division of the result by the positive real number $\sqrt{x_0x_0 + x_1x_1}\sqrt{y_0y_0 + y_1y_1}$.

For example, the angle between two lines l and m of $\mathbf{T}_2$ is simply the angular difference between their directions, that is, $\mathrm{dir}(m) \hat{-} \mathrm{dir}(l)$. If $l = \langle l^0, l^1, l^2 \rangle$ and $m = \langle m^0, m^1, m^2 \rangle$, we have $\mathrm{dir}(l) = [0, l^2, -l^1]$ and $\mathrm{dir}(m) = [0, m^2, -m^1]$, and therefore

$$\mathrm{ang}(l, m) = [\,0,\ m^2l^2 + m^1l^1,\ m^2l^1 - m^1l^2\,]. \tag{23}$$

This formula can also be derived by computing the angular difference between normals, instead of directions. Note that the angle between two lines of $\mathbf{T}_2$ as defined above ranges over all points of Ω_2. In higher-dimensional spaces, however, the angle between two intersecting lines is ambiguous, because there is no consistent way to orient the plane containing them. That means we cannot distinguish an angle from its negative. Given two directions $x = [0, x_1, \ldots x_\nu]$ and $y = [0, y_1, \ldots y_\nu]$, the best we can do is compute the co-sine of the angle between them,

$$\cos \mathrm{ang}(x, y) = \frac{x_1y_1 + \cdots + x_\nu y_\nu}{\sqrt{x_1x_1 + \cdots + x_\nu x_\nu}\sqrt{y_1y_1 + \cdots + y_\nu y_\nu}}. \tag{24}$$

For consistency, we may want to represent this angle as the point $[\,0,\ c,\ \sqrt{1 - c^2}\,]$ of Ω_2, where $c = \cos \mathrm{ang}(x, y)$. Note that this angle is always between 0 and π radians, inclusive.

6. Non-Euclidean geometries

In the same spirit, we can use oriented projective geometry to emulate oriented versions of the classical non-Euclidean geometries. For hyperbolic geometry, we use a two-sided version of Beltrami's model, consisting of all flats that intersect the unit disk of the front and back ranges. That is, we take all points $[x_0, .. x_\nu]$ such that $x^1x^1 + \cdots + x_\nu x_\nu \leq x_0 x_0$, and the flats obtained by joining those points. See figure 12.

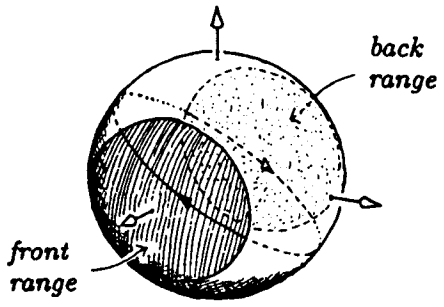

Figure 12. The two-sided hyperbolic plane.

The hyperbolic maps are then defined as the projective maps of $\mathbf{T}_2$ that take this set to itself. Those maps can be used to define hyperbolic congruence and perpendicularity, and hyperbolic metrics for distances and angles.

For elliptic geometry, we use the whole $\mathbf{T}_2$, and define the elliptic maps as those that preserve the standard polarity $\perp$. As in the hyperbolic case, those maps can be used to define the distance metric of elliptic geometry, which is simply the great-circle distance on the spherical model of $\mathbf{T}_2$. In fact, measuring angles in the Euclidean space $\mathbf{E}_\nu$ is equivalent to measuring distances on Ω_ν, viewed as the elliptic space of dimension $\nu - 1$. The equivalence is made evident by the map $[0, x_1, .. x_\nu] \mapsto [x_1, .. x_\nu]$, which takes Ω_ν to $\mathbf{T}_{\nu-1}$. These representations are reasonably efficient, and most of the data structures and operations of $\mathbf{T}_\nu$ (join, meet, projective transformations, etc.) can be used for non-Euclidean geometry without modification.

7. Final remarks

The summary of the last three chapters is that it is possible and convenient to emulate a variety of geometrical spaces in the two-sided space $\mathbf{T}_\nu$, in a way that retains the main advantages of the latter, such as the ability to distinguish antipodal directions and plus infinity from minus infinity. There are still many elementary geometric problems we can't discuss for lack of space (such as, for example, computing

angles and distances between arbitrary flats). Hopefully, the reader will have little trouble in figuring out the necessary formulas on his or her own.

Chapter 18
Representing flats by simplices

The analytic model of two-sided spaces has so far been limited to the representation of points (by homogeneous coordinates) and hyperplanes (by homogeneous coefficients). In this chapter we are going to consider one possible extension of these concepts to flats of arbitrary rank, namely the *simplex representation.* As one might expect, this representation has an analog in the unoriented framework. Compared to that, the handling of orientations requires no additional arithmetic: we have only to pay a bit more attention to the signs of coordinates and to the order of operands.

1. The simplex representation

A straightforward way to encode a flat a with rank $k \in \{2..n-2\}$ is the *simplex representation*, consisting of a $k \times n$ real matrix

$$\begin{pmatrix} a_0^0 & a_1^0 & \dots & a_\nu^0 \\ \vdots & & & \vdots \\ a_0^\kappa & a_1^\kappa & \dots & a_\nu^\kappa \end{pmatrix} \tag{1}$$

whose rows are the homogeneous coordinates of the vertices of any positive simplex of a. As we saw in chapter 4, two simplices are equivalent (determine the same flat) if and only if their matrices a and b satisfy $a = Kb$, where K is some $k \times k$ matrix with positive determinant. In other words, the flats of $\mathbf{T}_\nu$ with rank k can be identified with the equivalence classes of $k \times n$ matrices under this relation.

I will denote the class of all matrices equivalent to (1) by using square brackets instead of round ones. That is, for any proper simplex $a^0,..a^\kappa$, I will write

$$a^0 \vee a^1 \vee \dots \vee a^\kappa = \begin{bmatrix} a_0^0 & a_1^0 & \dots & a_\nu^0 \\ \vdots & & & \vdots \\ a_0^\kappa & a_1^\kappa & \dots & a_\nu^\kappa \end{bmatrix}$$

where $a^i = [a_0^i,..a_\nu^i]$. I will call any member of this class a *coordinate matrix* for the flat $a^0 \vee \dots \vee a^\kappa$.

In particular, the universe Υ of $\mathbf{T}_\nu$ is represented by the class of $n \times n$ matrices with positive determinant, and its opposite $\neg\Upsilon$ by those with negative determinant. The vacua Λ and $\neg\Lambda$ must be handled as special cases, since the obvious representation (a $0 \times n$ matrix) doesn't distinguish between them.

Of course, the simplex representation can also represent unoriented flats. We have only to regard two coordinate matrices equivalent if they are related by a $k \times k$ matrix with *nonzero* determinant.

2. Manipulating the simplex representation

2.1. Simplex orientation

Let $s = (s^0; .. s^\nu)$ be a simplex of $\mathbf{T}_\nu$ with n vertices. The orientation of s is given by the sign of the determinant of its coordinate matrix. That is,

$$\operatorname{sign}(s^0, .. s^\nu) = \operatorname{sign} \begin{vmatrix} s_0^0 & s_1^0 & \cdots & s_\nu^0 \\ s_0^1 & s_1^1 & \cdots & s_\nu^1 \\ \vdots & & & \vdots \\ s_0^\nu & s_1^\nu & \cdots & s_\nu^\nu \end{vmatrix}$$

Many interesting properties of the sign function follow immediately from this definition. For one thing, transposing any two vertices of a simplex reverses its sign, as does replacing any vertex by its antipode. A cyclic permutation of all vertices preserves the sign of the simplex if the rank n is odd, and reverses it if n is even. This is because a cyclic permutation of n objects is equivalent to $n - 1$ transpositions.

In particular, if $(p; q; r)$ is a positive triangle of $\mathbf{T}_2$, so are its cyclic permutations $(q; r; p)$ and $(r; p; q)$. On the other hand, if $(p; q; r; s)$ is a positive tetrahedron of $\mathbf{T}_3$, then $(q; r; s; p)$ and $(s; p; q; r)$ are negative tetrahedra, whereas $(r; s; p; q)$ is a positive one.

2.2. Join, meet, and relative orientation

Computing the join of two flats in the simplex representation is quite easy: we simply stack the coordinate matrix of one on top of that of the other (provided their ranks add to at most n).

However, other operations are substantially harder. To check whether two matrices s, t represent the same flat, we have to test whether there is a $k \times k$ real matrix A with positive determinant such that $s = At$. To compute $a^\vdash$ or $a^\dashv$ we have to find a basis for the orthogonal complement of the row space of the matrix of a. To compute the meet of two flats, we have to find a suitably oriented basis for the

intersection of the row spaces of their matrices. To test the relative orientation of two flats, we have to compute the determinant of the $n \times n$ matrix resulting from their join.

These computations can be carried out in practice by Gaussian elimination and similar numerical methods, with roughly $O(nk^2)$ running time. The algorithms are identical to those of unoriented projective geometry, except that we must be careful to preserve the orientation implicit in the order of the rows. For example, when swapping two rows of a simplex we must multiply one of them by -1.

3. The dual simplex representation

Recall that an hyperplane h of $\mathbf{T}_\nu$ can be represented by n homogeneous coefficients h^j, such that for any point $x = [x_0, .. x_\nu]$ we have

$$x \diamond h = \text{sign}(x_0 h^0 + x_1 h^1 + \cdots + x_\nu h^\nu). \tag{2}$$

Hyperplane coefficients are the basis of the *dual simplex representation* for flats of arbitrary rank. The idea is to represent a flat a of $\mathbf{T}_\nu$ with co-rank k by the coefficients of k hyperplanes $h_0, .. h_\kappa$ whose meet, in that order, is the flat a. It is convenient to write those numbers as an $n \times k$ *coefficient matrix*

$$\begin{pmatrix} a_0^0 & \cdots & a_\kappa^0 \\ a_0^1 & & a_\kappa^1 \\ \vdots & & \vdots \\ a_0^\nu & \cdots & a_\kappa^\nu \end{pmatrix} \tag{3}$$

where column j gives the coefficients of the jth hyperplane.

Observe that in this case we have $a^\dashv = (a_0)^\dashv \vee \cdots \vee (a_\kappa)^\dashv$. The homogeneous coordinates of the point $(a_i)^\dashv$ are the coefficients of the hyperplane a_i, viewed as a row vector: $(a_i)^\dashv = [a_i^0, .. a_i^\nu]$. It follows that the (primal) simplex representation of the flat $a^\dashv$ is the $k \times n$ matrix whose rows are the columns of (3), in the same order. We conclude that *a coefficient matrix for a flat a is the transpose of a coordinate matrix for the flat $a^\dashv$.*

From this it follows that two coefficient matrices a, b denote the same flat of $\mathcal{F}_n^{n-k}$ if and only if there is a $k \times k$ matrix K with positive determinant such that $b = aK$. I will denote the class of coefficient matrices equivalent to (3) by horizontal

square brackets above and below the matrix. That is, I will write

$$a_0 \wedge a_1 \wedge \cdots \wedge a_\kappa = \left[\begin{matrix} a_0^0 & \ldots & a_\kappa^0 \\ a_0^1 & & a_\kappa^1 \\ \vdots & & \vdots \\ a_0^\nu & \ldots & a_\kappa^\nu \end{matrix}\right]$$

where $a_i = \langle a_i^0, .. a_i^\nu \rangle$. As expected from duality, the positive vacuum of $\mathbf{T}_\nu$ is represented by the class of $n \times n$ coefficient matrices with positive determinant, and the negative vacuum by those with negative determinant. The universes $+\Upsilon$ and $-\Upsilon$ must be handled as a special case, since their coefficient matrices have zero columns.

Note that every point of a flat is on every hyperplane containing that flat. We conclude that if A is the $k \times n$ coordinate matrix of a flat, and B is its $n \times (n-k)$ coefficient matrix, then the product AB is the zero matrix of size $k \times (n-k)$. Conversely, if this is true of two full-rank matrices A and B, then they denote the same flat, except for orientation.

Symmetrically, the dual simplex representation makes it quite easy to compute the meet of two flats: it suffices to glue their coefficient matrices side by side. On the other hand, other operations (including join) require some variant of the Gaussian elimination algorithm.

3.1. Mixed representation

The simplex representation is also highly redundant, especially when k is close to n. The condition for equivalence stated above implies that the set of all flats of rank k in $\mathbf{T}_\nu$ has dimension $kn - k^2$, and yet their coordinate matrices have kn coefficients. In particular, a hyperplane has $n(n-1)$ coordinates, even though it can be represented by only n coefficients.

To avoid this high storage cost, we can use a *mixed representation*, in which a flat is represented by either a simplex or a dual simplex, whichever is smaller. That is, we use the coordinate matrix if the rank k is $< n/2$, and the coefficient matrix when $k > n/2$ (when $k = n/2$ we can use either form). This mixed strategy reduces the wasted storage, but doesn't eliminate it completely. The maximum waste now occurs with flats of intermediate rank $k \approx n/2$, which have $\approx n/2$ coordinates or coefficients, but only $\approx n^2/4$ degrees of freedom. One advantage of this strategy is that the polar complement operations $\vdash$ and $\dashv$ become trivial.

3.2. Converting between coordinates and coefficients

It is often necessary to compute the dual matrix of a flat from the primal one, or vice-versa. In particular, if we use the mixed representation described above, we will often have to do this conversion as part of join or meet operations. As discussed above, this problem can be reduced to that of computing the coordinate matrix of $a^{\vdash}$ given that of a. This can be done by Gaussian elimination.

4. The reduced simplex representation

A more promising way to reduce storage costs and the ambiguity of the matrix representation is to represent each flat by some "canonical" simplex. If we choose canonical simplices with matrices of a particular simple form, we can usually encode the latter with far less than kn elements.

In particular, we may consider the *reduced simplices* whose coordinate matrices have the form described below. I say that a $k \times n$ matrix a is *reduced* if there are k integers $0 \leq j_0 < j_1 < \cdots < j_\kappa \leq \nu$ (the *pivot columns*) such that, for all i,

(i) $a^i_{j_i}$ is ± 1 if $i = 0$, and 1 if $i > 0$.

(ii) $a^i_{j_i}$ is the only nonzero element in column j_i,

(iii) $a^i_{j_i}$ is the first nonzero element in row i, for all i.

For example, here is a reduced simplex of rank 4 in $\mathbf{T}_9$:

$$\begin{array}{cccccccccc} & j_0 & & & j_1 & & j_2 & j_3 & & \\ & \downarrow & & & \downarrow & & \downarrow & \downarrow & & \\ \left(\right. 0 & 1 & \boxed{3} & \boxed{-6} & 0 & \boxed{4} & 0 & 0 & \boxed{5} & \boxed{-1} \left. \right) \\ 0 & 0 & 0 & 0 & 1 & \boxed{3} & 0 & 0 & \boxed{3} & \boxed{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \boxed{-6} & \boxed{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \boxed{9} & \boxed{1} \end{array} \tag{4}$$

Conditions (i) and (ii) say that the $k \times k$ submatrix of a formed by columns $j_0, .. j_\kappa$ is the identity matrix, except that its first element may be -1. Condition (iii) implies that any other matrix equivalent to a that satisfies (i) and (ii) will have a sequence of pivot columns that is lexicographically larger than $j_0, .. j_\kappa$. It follows that there is at most one reduced matrix equivalent to a given one, or, in other words, that every flat contains at most one reduced simplex.

On the other hand, we can transform any given simplex to an equivalent reduced one, by a straightforward variant of the Gaussian elimination algorithm. The

only novelty is that during the algorithm we must watch for operations that reverse the orientation of the simplex (namely, swapping of two rows and multiplication of a row by a negative number), and compensate for them by also negating the first row.

This shows we can represent an arbitrary flat of $\mathbf{T}_\nu$ with rank k by a triplet (J, R, s), where s is a sign bit giving the value of $a^0_{j_0}$, J is the set of pivot columns, and R is a linear array containing the remaining variable elements of the reduced simplex, namely a^i_j for $j > i, j \notin \mathrm{J}$. These elements are shown boxed in example (4). This information occupies at most $k(n-k)$ floating-point words plus $n+1$ bits of storage. (Again, we are not counting the space needed to store the ranks k and n.)

The operations of meet, join, and relative complement can be computed by reconstituting the operands to the full simplex representation, computing a simplex for the result, and reducing it as described above. It is possible to combine all those steps and get algorithms that work directly with the reduced operands and reduce the result on-the fly; however, the relatively modest savings in space and arithmetic operations one would obtain this way must be balanced against the cost of increased program complexity.

Chapter 19
Plücker coordinates

Plücker (or *Grassmann*) *coordinates* are another way of extending homogeneous coordinates to flats of arbitrary rank. Compared to the various flavors of simplex representation, Plücker coordinates have the advantage of being mathematically more elegant, and the disadvantage of requiring more storage and computer time. Therefore, Plücker coordinates are valuable mostly in theoretical work, and in computations restricted to two- and three-dimensional space.

1.1. The Plücker coordinates of a line

In chapter 7 we saw that a flat a of rank r is uniquely determined if we know its orientation relative to all flats of co-rank r. For example, a line l of $\mathbf{T}_3$ is uniquely determined by the values of $l \diamond h$ when h ranges over all lines of $\mathbf{T}_3$. Let $(u;v)$ and $(x;y)$ be simplices spanning l and h, respectively. The value of $l \diamond h$ is then given by

$$l \diamond h = \text{sign} \begin{vmatrix} u_0 & u_1 & u_2 & u_3 \\ v_0 & v_1 & v_2 & v_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{vmatrix}.$$

We can expand this determinant into a sum of six terms, each being the product of a 2×2 minor from the first two rows, and the "complementary" minor from the bottom two rows. That is,

$$l \diamond h = \text{sign}\big(l_{\{01\}} h_{\{23\}} - l_{\{02\}} h_{\{13\}} + l_{\{12\}} h_{\{03\}} + l_{\{03\}} h_{\{12\}} - l_{\{13\}} h_{\{02\}} + l_{\{23\}} h_{\{01\}} \big)$$

where

$$l_{\{ij\}} = \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} \quad \text{and} \quad h_{\{ij\}} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}.$$

It follows that the line l is uniquely determined by the six minors $l_{\{01\}}, l_{\{02\}}, \ldots, l_{\{23\}}$. These are the *Plücker coordinates* of the line.

Observe that the choice of the simplex $(u;v)$ affects the Plücker coefficients of l only by a positive factor. That is, if $(p;q) = A(u;v)$ for some 2×2 matrix A with positive determinant, then the Plücker coefficients $m_{\{ij\}}$ computed from $(p;q)$ will satisfy $m_{\{ij\}} = |A|\, l_{\{ij\}}$. For example, the line l determined by the simplex

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 2 & 1 & 0 \end{pmatrix} \tag{1}$$

has Plücker coordinates

$$l_{\{01\}} = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 \qquad l_{\{02\}} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad l_{\{12\}} = \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = 2$$

$$l_{\{03\}} = \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0 \qquad l_{\{13\}} = \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} = -6 \qquad l_{\{23\}} = \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -3.$$

This six-number representation of lines doesn't seem to be well known in the computer graphics community. One notable exception is Patrick Hanrahan's geometry calculator[1]. Lines are commonly represented as a pair (point, direction vector), as a pair of points, or as the intersection of two planes. In three dimensions, all those schemes are generally less elegant and computationally less efficient than Plücker coordinates.

1.2. Plücker coordinates for general flats

The discussion that led us to the Plücker coordinates of a line is equally valid for a flat a of arbitrary rank k in any space $\mathbf{T}_\nu$. By the separation theorem, a is uniquely determined by the values of $a \diamond h$ where h ranges over all flats of $\mathbf{T}_\nu$ with rank $m = n - k$. Therefore, if s is any positive simplex of a, and x a positive simplex of h, then

$$a \diamond h \;=\; \text{sign} \begin{vmatrix} s_0^0 & s_1^0 & \dots & \dots & s_\nu^0 \\ \vdots & \vdots & & & \vdots \\ s_0^\kappa & s_1^\kappa & \dots & \dots & s_\nu^\kappa \\ x_0^0 & x_1^0 & \dots & \dots & x_\nu^0 \\ \vdots & \vdots & & & \vdots \\ x_0^\mu & x_1^\mu & \dots & \dots & x_\nu^\mu \end{vmatrix}. \tag{2}$$

We can expand the determinant (2) into a sum of terms, each being the product of a $k \times k$ minor determinant from the first k rows, and an $m \times m$ minor determinant

from the last m rows:

$$a \diamond h = \sum_{\substack{I \cap J = \emptyset \\ I \cup J = \{0..\nu\} \\ |I|=k, |J|=m}} (-1)^{|I>J|} a_I h_J \tag{3}$$

where

$$a_{\{i_0,i_1,\ldots,i_\kappa\}} = \begin{vmatrix} s^0_{i_0} & \cdots & s^0_{i_\kappa} \\ \vdots & & \vdots \\ s^\kappa_{i_0} & \cdots & s^\kappa_{i_\kappa} \end{vmatrix} \tag{4}$$

and

$$h_{\{j_0,j_1,\ldots j_\mu\}} = \begin{vmatrix} x^0_{j_0} & \cdots & x^0_{j_\mu} \\ \vdots & & \vdots \\ x^\mu_{j_0} & \cdots & x^\mu_{j_\mu} \end{vmatrix}. \tag{5}$$

The exponent $|I > J|$ in formula (3) is the number of pairs $i > j$ with $i \in I$ and $j \in J$. (The latter is simply the number of inversions in the permutation of $\{0..\nu\}$ that consists of all elements of I followed by those of J.)

From formula (3) we conclude that a is uniquely characterized by the $\binom{n}{k}$ minor determinants a_I. Those numbers are by definition the *Plücker* or *homogeneous coordinates* of a. The set $\{i_0, i_1, \ldots, i_\kappa\}$ of the columns included in the determinant (4) is the *label* of that coordinate. The labels range over all k-element subsets of $N = \{0,..\nu\}$.

From linear algebra we know that the determinants (4) are all zero if and only if the rows of the matrix

$$\begin{pmatrix} s^0_0 & s^0_1 & \cdots & s^0_\nu \\ \vdots & & & \vdots \\ s^0_0 & s^0_1 & \cdots & s^0_\nu \end{pmatrix}$$

are linearly dependent, that is, if the simplex s is degenerate. Therefore, the Plücker coordinates of a flat are not all zero.

As in the case of lines of $\mathbf{T}_3$, the particular choice of the representative simplex $s^0,..s^\kappa$ (and any scaling of the homogeneous coordinates of each s^i) affects the Plücker coordinates of a only to the extent of multiplying all of them by some positive real number. We conclude that *two flats of* $\mathbf{T}_\nu$ *are the same flat if and only if their Plücker coordinates differ only by a positive factor.*

Because of this scale ambiguity, it doesn't make sense to refer to the magnitude of a Plücker coordinate in isolation, but only to a complete set of them. We might consider normalizing the coordinates in some way (e.g., so that their squares

add to 1); however, as in the case of point coordinates, this normalization is usually not worth the cost.

1.3. The natural order of coordinates

In formulas and in computer programs it is more convenient to write the homogeneous coordinates of a flat in some canonical order, so that the labels can be omitted. A convenient choice is to enumerate the labels in increasing order of their *binary value.*

By definition, the binary value of a finite set X of natural numbers is $\operatorname{binv} X = \sum_{x \in X} 2^x$. Note that if we write $\operatorname{binv} X$ in base two, then the elements of X are the positions of the "1" bits, from right to left. For example,

$$\begin{array}{c} \scriptstyle 6\ 4\,3\ \ 0 \\ \scriptstyle \downarrow\ \downarrow\downarrow\ \ \downarrow \\ \operatorname{binv}\{0\,3\,4\,6\} = 1011001_2 = 89_{10}. \end{array}$$

When dealing with Plücker coordinates we have often to consider the collection of all subsets of $\mathbf{N}$ with a fixed size k. I will denote by $k{:}i$ the ith subset of this list, in order of increasing binary value, starting from $i = 0$. If $k{:}i = X$, I will call i the *index* of X, and write $i = \#X$. Note that $\#X$ is usually different from (and much smaller than) $\operatorname{binv} X$. For example, the list of the three-element subsets of $\mathbf{N}$ and their binary values are

3:0	3:1	3:2	3:3	3:4	3:5	3:6	3:7	3:8	3:9	3:10	···
{0 1 2}	{0 1 3}	{0 2 3}	{1 2 3}	{0 1 4}	{0 2 4}	{1 2 4}	{0 3 4}	{1 3 4}	{2 3 4}	{0 1 5}	···
7	11	13	14	19	21	22	25	26	28	35	···

In general, a set I precedes another set J in this ordering if $\max I < \max J$, or $\max I = \max J$ and $I \setminus \{\max I\}$ precedes $J \setminus \{\max J\}$. We can obtain this ordering also by writing each subset as a *decreasing* sequence of numbers, and sorting those sequences in *increasing* lexicographic order. Note that for any j all subsets of $\{0\,..\,j\}$ occur together at the beginning of the list, and before any subset involving elements greater than j.

I will use the notation $[z_0, z_1, \ldots]^k$ for the flat of $\mathbf{T}_\nu$ with rank k whose Plücker coordinates are $z_0, z_1, \ldots$, listed in the natural order of their labels. Therefore, the label set of z_i is $k{:}i$. For example, I will denote the line of example (1) as

$$\begin{array}{c} l = [\ 2,\ \ 1,\ \ 2,\ \ 0,\ \ -6,\ \ -3\ ]^2. \\ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow \\ \{0\,1\}\ \{0\,2\}\ \{1\,2\}\ \{0\,3\}\ \{1\,3\}\ \{2\,3\} \end{array}$$

With this convention, the uniqueness and equivalence properties of Pücker coordinates can be restated as follows: *For any k, $[x_0, x_1, \ldots]^k$ and $[y_0, y_1, \ldots]^k$ are the same flat if and only if $x_i = \alpha y_i$ for some $\alpha > 0$ and all i.*

1.4. Points and hyperplanes

Note that when $k = 1$ this notation coincides with the one we have been using for plain homogeneous coordinates. That is, the point with Plücker coordinates $[u_0, u_1, \ldots]^1$ is the point with homogeneous coordinates $[u_0, u_1, \ldots]$.

A hyperplane or $\mathbf{T}_\nu$ has n Plücker coordinates, whose labels are the sets $\{0,..\nu\} \setminus i$ for each i. These coordinates are the coefficients of the hyperplane as commonly used in graphics programming, except that they are listed in reverse order and half the signs are reversed. We will examine this in more detail later on.

1.5. Vacuum and universe

The universe of $\mathbf{T}_\nu$ and its antipode have a single coordinate, whose label is the entire set $\{0..\nu\}$. It is easy to see that $\Upsilon\nu = [+1]^n$, and $\neg\Upsilon\nu = [-1]^n$. By convention, the flats of zero rank (the vacua) also have a single coordinate, whose label is the empty set and whose sign describes the flat's orientation. That is, $\Lambda = [+1]^0$ and $\neg\Lambda = [-1]^0$.

1.6. Lines in the plane

A line l of $\mathbf{T}_2$ has three Plücker coordinates, namely $l_{\{01\}}, l_{\{02\}}, l_{\{12\}}$. If $(u;v)$ is a positive simplex of l with $u = [u_0, u_1, u_2]$ and $v = [v_0, v_1, v_2]$, then

$$l_{\{01\}} = \begin{vmatrix} u_0 & u_1 \\ v_0 & v_1 \end{vmatrix} \qquad l_{\{02\}} = \begin{vmatrix} u_0 & u_2 \\ v_0 & v_2 \end{vmatrix} \qquad l_{\{12\}} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

Here are some lines of $\mathbf{T}_2$, and their Plücker coordinates:

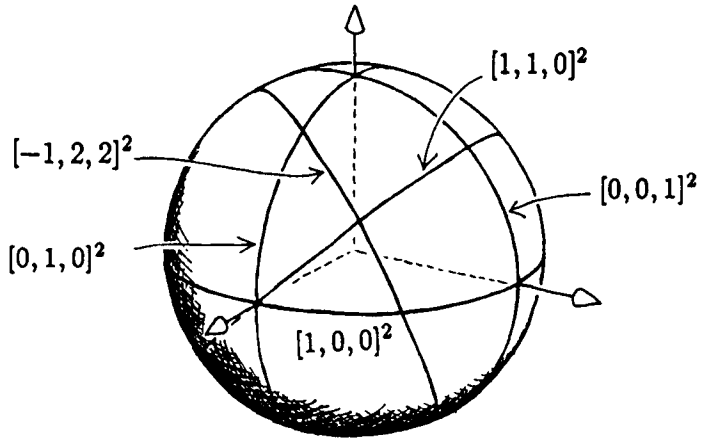

Figure 1. Some lines of $\mathbf{T}_2$ and their Plücker coordinates.

$$\begin{aligned}
\mathbf{e}^0 \vee \mathbf{e}^1 &= [1,0,0]^2 && \text{the } x\text{-axis} \\
\mathbf{e}^0 \vee \mathbf{e}^2 &= [0,1,0]^2 && \text{the } y\text{-axis} \\
\mathbf{e}^1 \vee \mathbf{e}^2 &= [0,0,1]^2 && \Omega \\
\mathbf{e}^0 \vee [1,1,1] &= [1,1,0]^2 && \text{bisector of first quadrant} \\
[1,1,0] \vee [1,0,2] &= [-1,2,2]^2 && \text{see figure 1.}
\end{aligned}$$

1.7. Canonical flats

Let $\mathrm{I} = \{i_0, i_1, \ldots, i_\kappa\}$ be any subset of $\mathrm{N} = \{0,..\nu\}$ with size k and $i_0 < i_1 < \cdots < i_\kappa$. By definition, the join of the canonical points $\mathbf{e}^{i_0}, .. \mathbf{e}^{i_\kappa}$, in that order, is the *canonical flat of* $\mathbf{T}_\nu$ *with label* I, denoted by $\mathbf{e}^{\mathrm{I}}$. Its Plücker coordinates are all zero, except for $(\mathbf{e}^{\mathrm{I}})_{\mathrm{I}} = 1$.

For example, the canonical lines of $\mathbf{T}_3$ are

$$\begin{aligned}
\mathbf{e}^{\{0\,1\}} = \mathbf{e}^0 \vee \mathbf{e}^1 &= [1\,0\,0\,0\,0\,0]^2 && x\text{-axis} \\
\mathbf{e}^{\{0\,2\}} = \mathbf{e}^0 \vee \mathbf{e}^2 &= [0\,1\,0\,0\,0\,0]^2 && y\text{-axis} \\
\mathbf{e}^{\{1\,2\}} = \mathbf{e}^1 \vee \mathbf{e}^2 &= [0\,0\,1\,0\,0\,0]^2 && \text{line at infinity of the } xy \text{ plane} \\
\mathbf{e}^{\{0\,3\}} = \mathbf{e}^0 \vee \mathbf{e}^3 &= [0\,0\,0\,1\,0\,0]^2 && z\text{-axis} \\
\mathbf{e}^{\{1\,3\}} = \mathbf{e}^1 \vee \mathbf{e}^3 &= [0\,0\,0\,0\,1\,0]^2 && \text{line at infinity of the } xz \text{ plane} \\
\mathbf{e}^{\{2\,3\}} = \mathbf{e}^2 \vee \mathbf{e}^3 &= [0\,0\,0\,0\,0\,1]^2 && \text{line at infinity of the } yz \text{ plane.}
\end{aligned}$$

2. The canonical embedding

Recall that the canonical embedding of $\mathbf{T}_\nu$ into a space $\mathbf{T}_\mu$ with $\mu > \nu$ is obtained by appending $\mu - \nu$ zeros to the homogeneous coordinates of every point. In general, if a is a flat of rank k in $\mathbf{T}_\nu$, its canonical embedding $\hat{a}$ is obtained by appending $\mu - \nu$ zero columns at the right end of its coordinate matrix:

$$\begin{pmatrix} s_0^0 & s_1^0 & \cdots & s_\nu^0 \\ \vdots & & & \vdots \\ s_0^0 & s_1^0 & \cdots & s_\nu^0 \end{pmatrix} \mapsto \begin{pmatrix} s_0^0 & s_1^0 & \cdots & s_\nu^0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ s_0^0 & s_1^0 & \cdots & s_\nu^0 & 0 & \cdots & 0 \end{pmatrix}.$$

What is the effect of this embedding in terms of Plücker coordinates? The flat $\hat{a}$ has $\binom{m}{k}$ coordinates, each labeled with a subset of $\{0..\mu\}$ with size k. Now observe that in the natural order of these sets, all those that are subsets of $\{0..\nu\}$ occur together at the beginning of the list. Therefore, the first $\binom{n}{k}$ Plücker coordinates

of $\hat{a}$ are exactly the coordinates of a. Moreover, any other coordinate of $\hat{a}$ is zero, since it is the determinant of a $k \times k$ matrix that includes at least one zero column. We conclude that *the canonical embedding of* $\mathbf{T}_\nu$ *into* $\mathbf{T}_\mu$ *merely appends* $\binom{m}{k} - \binom{n}{k}$ *zeros to the Plücker coordinates of every flat of rank* k.

3. Plücker coefficients

The *Plücker coefficients* of general flats are related to the homogeneous coefficient of hyperplanes in the same way that Plücker coordinates are related to point coordinates. If a is a flat with coefficient matrix

$$\begin{bmatrix} h_0^0 & \dots & h_\kappa^0 \\ h_0^1 & & h_\kappa^1 \\ \vdots & & \vdots \\ h_0^\nu & \dots & h_\kappa^\nu \end{bmatrix}$$

(that is, $a = h_0 \wedge \cdots \wedge h_\kappa$, where $h_j = \langle h_j^0, .. \rangle$ is the hyperplane $h_j^\nu\rangle$), then the Plücker coefficients of a are by definition the $k \times k$ minor determinants of that matrix:

$$a^{\{j_0, j_1, \dots, j_\kappa\}} = \begin{vmatrix} h_0^{j_0} & \dots & h_0^{j_\kappa} \\ \vdots & & \vdots \\ k_\kappa^{j_0} & \dots & h_\kappa^{j_\kappa} \end{vmatrix}.$$

As in the case of coordinates, it is convenient to list the Plücker coefficients of a in the natural order of their label sets. I will denote the ith element of this list (counting from 0) by a^i; i is the *index* of the coefficient. I will also write $\langle c^0, c^1, \dots \rangle^k$ to denote the flat of co-rank k with Plücker coefficients $c^0, c^1, \dots$. Like the Plücker coordinates, the Plücker coefficients are unique only up to a positive scaling factor.

The Plücker coefficients of a flat a are distinct from but closely related to the Plücker coordinates of a. As we shall see in the next chapter, to convert from one representation to the other we have to reverse the order of all elements, and negate some of them. In particular, a line of $\mathbf{T}_2$ with coefficients $\langle a, b, c \rangle$ has Plücker coordinates $[c, -b, a]^2$, and conversely.

4. Storage efficiency

The Plücker representation for a rank k flat of $\mathbf{T}_\nu$ requires $\binom{n}{k}$ coordinates, versus the kn required by the simplex representation. Obviously, Plücker coordinates are far too expensive for large values of k and n. For $\nu \leq 4$, however, they are no more expensive than the simplex form, as shown below:

rank of space $= n$	2	3		4			5			
rank of flat $= k$	1	1	2	1	2	3	1	2	3	4
Simplex $= \min\{kn, (n-k)n\}$	2	3	3	4	8	4	5	10	10	5
Plücker $= \binom{n}{k}$	2	3	3	4	6	4	5	10	10	5
Reduced simplex $= k(n-k)+1$ †	2	3	3	4	5	4	5	7	7	5
$\dim \mathcal{F}_n^k = k(n-k)$	1	2	2	3	4	3	4	6	6	4

(6)

† Assuming the sign bit and the pivot indices together use no more space than one matrix element.

As the table shows, for two-, three-, and four-dimensional geometry the Plücker coordinate representation is no bigger than the (unreduced) simplex representation. In fact, it is slightly smaller for lines in three-space (six numbers instead of eight). The reduced simplex form is somewhat more economical than the Plücker one, but it is not clear whether that is enough to offset its other drawbacks.

5. The Grassmann manifolds

According to the homogeneous model, the set $\mathcal{F}_n^k$ of all rank k flats of $\mathbf{T}_\nu$ is also the set of all k-dimensional oriented linear subspaces of $\mathbf{R}^n$. This set is the *oriented Grassmann manifold.*[2:*VII,XIV*]

From the simplex representation, we know that every element of $\mathcal{F}_n^k$ is an equivalence class of $\mathbf{R}^{k\times n}$ (the $k \times n$ matrices), where two matrices are equivalent if one is obtained from the other through multiplication by an $k \times k$ matrix with positive determinant. The set of these matrices has dimension k^2. From these observations it follows eventually that $\mathcal{F}_n^k$ is a manifold of dimension $kn - k^2 = k(n-k)$. These numbers are listed as the bottom row of table (6).

Notice that in general there is a wide gap between the dimension of $\mathcal{F}_n^k$ and the number of coordinates used by the simplex and Plücker representations. This gap is already evident for lines in three-space ($k = 2$, $n = 4$): the set of all such lines is only a four-dimensional manifold, but each line has six Plücker coordinates. One of these six degrees of freedom is "wasted" by equivalence under positive scaling. The other is lost because not all sextuples $l_0, .. l_5$ of real numbers are the Plücker

coordinates of some line. In fact, this happens if and only if the numbers satisfy the equation

$$l_0 l_5 - l_1 l_4 + l_2 l_3 = 0. \tag{7}$$

In general, $\binom{n}{k}$ real numbers $z_0, z_1, \ldots$ can be interpreted as the Plücker coordinates of a flat of rank k of $\mathbf{T}_\nu$ if and only if they are not all zeros, and they satisfy a number of equations of the form

$$\sum_{i,j} \lambda_{ijr} z_i z_j = 0$$

for $r = 0, 1, \ldots$, where the coefficients λ_{ijr} are in $\{-1, 0, +1\}$. For more details see for example the book by Hodge and Pedoe.[2:VII]

6. References

[1] P. Hanrahan: **A homogeneous geometry calculator.** Technical memo 3-D no. 7, Computer Graphics Laboratory, New York Inst. of Technology (September 1984).

[2] W. V. D. Hodge, D. Pedoe: **Methods of algebraic geometry.** Cambridge University Press (1952).

Chapter 20
Formulas for Plücker coordinates

Let's now examine the question of computing the basic operations of oriented projective geometry — join, meet, and polar complement — given the homogeneous coordinates of the operands, as defined in the previous chapter.

1. Join

According to the definition, the join of m points $p^0 \vee p^1 \vee \cdots \vee p^\mu$ is the flat whose Plücker coordinates are all the $m \times m$ minor determinants of the matrix

$$\begin{pmatrix} p_0^0 & p_1^0 & \cdots & \cdots & p_\nu^0 \\ \vdots & & & & \vdots \\ p_0^\mu & p_1^\mu & \cdots & \cdots & p_\nu^\mu \end{pmatrix}$$

Let us now compute the join of two arbitrary flats a and b of $\mathbf{T}_\nu$, given their Plücker coordinates. Let $r = \text{rank}(a)$, $s = \text{rank}(b)$, $t = r + s = \text{rank}(a \vee b)$ (we must of course have $t \leq n$ for the join to be defined). Let u be a positive simplex of a, and v be one of b. Let also $u^i = [u_0^i, u_1^i, \ldots, u_\nu^i]$, and $v^i = [v_0^i, \ldots, v_\nu^i]$ for all i. Then $(u^0; .. u^\kappa; v^0; .. v^\mu)$ is a positive simplex of $a \vee b$. The Plücker coordinates of $c = a \vee b$ are therefore the $t \times t$ minor determinants of the $t \times n$ matrix

$$C = \begin{pmatrix} u_0^0 & u_1^0 & \cdots & \cdots & u_\nu^0 \\ \vdots & & & & \vdots \\ u_0^\rho & u_1^\rho & \cdots & \cdots & u_\nu^\rho \\ v_0^0 & v_1^0 & \cdots & \cdots & v_\nu^0 \\ \vdots & & & & \vdots \\ v_0^\sigma & v_1^\sigma & \cdots & \cdots & v_\nu^\sigma \end{pmatrix}. \tag{1}$$

That is, the coordinates are the numbers

$$(a \vee b)_{\{k_0,..k_\tau\}} = c_{\{k_0,..k_\tau\}} = \begin{vmatrix} u^0_{k_0} & u^0_{k_1} & \dots & \dots & u^0_{k_\tau} \\ \vdots & & & & \vdots \\ u^\rho_{k_0} & u^\rho_{k_1} & \dots & \dots & u^\rho_{k_\tau} \\ v^0_{k_0} & v^0_{k_1} & \dots & \dots & v^0_{k_\tau} \\ \vdots & & & & \vdots \\ v^\sigma_{k_0} & v^\sigma_{k_1} & \dots & \dots & v^\sigma_{k_\tau} \end{vmatrix}$$

where $\{k_0,..k_\tau\}$ ranges over all sorted subsets of $\{0,..\nu\}$ of size t. We can expand each determinant c_{K} in terms of the $r \times r$ minors of the first r rows, and the $s \times s$ minors of the last s rows, according to the formula

$$(a \vee b)_{\mathrm{K}} = \sum_{\substack{\mathrm{I}\cup\mathrm{J}=\mathrm{K} \\ \mathrm{I}\cap\mathrm{J}=\emptyset \\ |\mathrm{I}|=r,|\mathrm{J}|=s}} (-1)^{|\mathrm{I}>\mathrm{J}|}\, a_{\mathrm{I}} b_{\mathrm{J}} \tag{2}$$

This formula is easily derived from the definition of determinants. Observe that the coordinates c_{K} of $a \vee b$ are sums of products of the form $a_{\mathrm{I}} b_{\mathrm{J}}$ with coefficients ± 1. The coordinates of the join are therefore bilinear homogeneous functions of those of its two operands.

2. Incidence

The formulas for join also give us a way to test whether a point lies on a flat, or whether two flats intersect. Recall that a point x is incident to a flat a if and only if $a \vee x = \mathbf{0}$ (provided we consider x and $-x$ to be incident to each other, too). In general, flats a and b intersect if and only if $a \vee b = \mathbf{0}$.

Algebraically, two flats a and b of $\mathbf{T}_\nu$ have a point in common if and only if the rows of the join matrix C in (1) are not linearly independent. In this case, the minor determinants c_{K} are all zero. This suggests we define the homogeneous coordinates of the null object $\mathbf{0}^k$ as being $[0, 0, \dots, 0]^k$.

With this convention, formula (2) will automatically return the correct value in all cases. In fact, the test $a \vee b = \mathbf{0}$ is a convenient way to check whether two flats a and b intersect, and in particular whether a point a lies on a flat b. In general, for two flats of ranks r and s in $\mathbf{T}_\nu$, this method tells us to evaluate $\binom{n}{r+s}$ bilinear functions of the coordinates, and check whether they are all zero.

In particular, a point $x = [x_0,..x_\nu]$ lies on a flat a if and only if $x \vee a = \mathbf{0}$,

that is

$$\sum_{i \in K} (-1)^{|i>K\setminus i|} x_i a_{K\setminus i} = 0 \tag{3}$$

for all $K \subseteq N$ with $|K| = \text{rank}(A) + 1$. For example, a point $x = [x_0, .. x_3]$ of $\mathbf{T}_3$ is on the line $a = [a_0, .. a_5]^2$ if and only if

$$\begin{cases} a_2x_0 - a_1x_1 + a_0x_2 = 0 \\ a_4x_0 - a_3x_1 + a_0x_3 = 0 \\ a_5x_0 - a_3x_2 + a_1x_3 = 0 \\ a_5x_1 - a_4x_2 + a_2x_3 = 0.. \end{cases} \tag{4}$$

As written above this test seems unnecessarily expensive, since it requires 12 multiplications, 8 additions, and 4 tests for zero. In contrast, if we represent a line as the intersection of two independent planes, we can test for incidence with 8 multiplications, 6 additions, and 2 tests for zero.

However, note that the linear equations (4) cannot be all independent. Since they are necessary and sufficient conditions for x to be on the line a, their solution space is two-dimensional. This means exactly two of the equations are redundant: we can write them as linear combinations of the other two, which must be linearly independent. The latter determine two distinct unoriented planes whose intersection is the line a.

So, when testing a point against a given line, it is advantageous to examine equations (4) first and select two independent ones. One way to do that is to look for a non-zero coordinate a_j, and pick the two equations where that coordinate appears as a coefficient. By inspection one can check that the two equations have the form

$$\begin{aligned} \cdots \pm a_j x_i \pm \cdots + 0 \cdot x_k \pm \cdots = 0 \\ \cdots + 0 \cdot x_i \pm \cdots \pm a_j x_k \pm \cdots = 0 \end{aligned}$$

and therefore are linearly independent. Therefore, the test for incidence reduces in the worst case to five zero tests to find a non-zero a_i, plus six multiplications, four additions, and two zero tests to check the two corresponding equations. If lines and points are randomly distributed in space (for most definitions of the word "random"), then on the average the incidence test will terminate after only three multiplications, two additions, and two tests for zero.

In general, if a is a flat of rank k in $\mathbf{T}_\nu$, the set of all vectors $x \in \mathbf{R}^n$ that satisfy the system of $\binom{n}{k+1}$ equations given by formula (3) is a subspace of dimension k. So, there are exactly $n-k$ linearly independent equations in that system which define $n - k$ hyperplanes whose intersection is a. Each of these hyperplanes is characterized by a label set $K \subseteq N$ with $k + 1$ elements; the ith coefficient of that

hyperplane is

$$\begin{cases} (-1)^{|i>K\setminus i|}\, a_{K\setminus i} & \text{if } i \in K, \\ 0 & \text{if } i \notin K. \end{cases} \tag{5}$$

It turns out that if the Plücker coordinate a_J is non-zero, the hyperplanes with label sets $J \cup \{i\}$, for each $i \in N \setminus J$, are independent. This is obvious once we realize that, among those $n-k$ hyperplanes, the one with label set $K = J \cup \{i\}$ is the only one with a non-zero coefficient for x_i. Notice also that each of the hyperplanes (5) has at most $k+1$ non-zero coefficients. Therefore, the point incidence test for a reduces to a search for a non-zero Plücker coordinate, followed by $(n-k)(k+1)$ multiplications, $(n-k)k$ additions, and $n-k$ tests for zero in the worst case. For random inputs, that reduces to a little more than $k+1$ multiplications, k additions, and one test for zero, on the average.

3. Relative orientation

The formula for $a \vee b$ becomes a bit simpler when the two flats have complementary ranks, that is, when $\text{rank}(a) + \text{rank}(b) = n$. In that case, the result is a flat of rank n: the universe Υ of $\mathbf{T}_n$, its opposite, or (if the two flats are not disjoint) the undefined flat $\mathbf{0}^n$.

A flat of rank n has only one Plücker coordinate c_N, where $N = \{0,..\nu\}$. Since positive scale factors do not matter, the only important property of that coordinate is its sign. We conclude that c_N is simply the relative orientation function $a \diamond b$. According to formula (2), its value is given by

$$c_N = \sum_{\substack{I\cup J=N \\ I\cap J=\emptyset \\ |I|=r,|J|=s}} (-1)^{|I>J|} a_I b_J$$

that is,

$$a \diamond b = \text{sign}\left(\sum_{\substack{K\subseteq N \\ |K|=r}} (-1)^{|K>\overline{K}|} a_K b_{\overline{K}} \right) \tag{6}$$

where $\overline{K}$ is the set complement of K relative to $N = \{0,..\nu\}$.

4. Polarity

The condition for two points of $\mathbf{T}_\nu$ to be polar is obviously that the dot product of their homogeneous coordinates be zero:

$$[x_0,..x_\nu] \perp [y_0,..y_\nu] \;\Leftrightarrow\; x_0y_0 + \cdots x_\nu y_\nu = 0.$$

In general, the condition for a point x to be polar to a flat a is

$$\sum_{i \in \overline{K}} (-1)^{|\{i\}>K|} x_i a_K = 0$$

for all $K \subseteq N$ with $|K| = \text{rank}(a)$.

4.1. Polar complement

Let a be a flat of $\mathbf{T}_\nu$ with rank r. Its polar complements in $\mathbf{T}_\nu$ are given by

$$(a^\vdash)_K = (-1)^{|\overline{K}>K|} a_{\overline{K}} \tag{7}$$

$$(a^\dashv)_K = (-1)^{|K>\overline{K}|} a_{\overline{K}} \tag{8}$$

where $\overline{K}$ denotes the complement of K with respect to N.

4.2. Meet

Formulas for the meet of two flats can be obtained by combining those for join and polar complement. From $a \wedge b = (a^\vdash \vee b^\vdash)^\dashv$ we get

$$(a \wedge b)_K = \sum_{\substack{I \cap J = K \\ I \cup J = N \\ |I|=r,\, |J|=s}} (-1)^{|\overline{J}>\overline{I}|} a_I b_J \tag{9}$$

4.3. Representative simplex

With the polar complement formulas above we are able to select a representative simplex from a flat b, given its Plücker coordinates. We have only to compute the polar complement $a = b^\vdash$ (formula (7)), then find a set of independent hyperplanes that contain the flat a (formula (8)), and finally list the polar complements of those hyperplanes.

5. Formulas for computers

In practice, a procedure that computes $A \vee B$ will be given the Plücker coordinates of the operands in natural order, as two arrays $[a_0, a_1 . .]$ and $[b_0, b_1, . .]$, and is expected to return the result C in the same format.

Since Plücker coordinates are practical only for spaces of dimension four or less, the best way to implement the basic geometric operations ($\vee$, $\wedge$, $\vdash$, etc.) is to write a separate routine for each combination of operand ranks, with summations expanded by hand. The resulting formulas are given in the tables below.

5.1. Formulas for one-dimensional geometry

In one dimension the only non-trivial flats are points. The join, meet, and relative orientation of two points are given by the same formula, but the polar complements $\vdash$ and $\dashv$ are distinct:

point $\leftarrow$ point$^{\vdash}$
$c_0 \leftarrow -a_1$ $c_1 \leftarrow a_0$

line $\leftarrow$ point $\vee$ point vacuum $\leftarrow$ point $\wedge$ point sign $\leftarrow$ point $\diamond$ point
$c_0 \leftarrow a_0 b_1 - a_1 b_0$

point $\leftarrow$ point$^{\dashv}$
$c_0 \leftarrow a_1$ $c_1 \leftarrow -a_0$

5.2. Formulas for two-dimensional geometry

In two dimensions the interesting flats are points and lines. The formulas for join of two points and meet of two lines are the same, and therefore can be computed by the same routine:

line $\leftarrow$ point $\vee$ point point $\leftarrow$ line $\wedge$ line
$c_0 \leftarrow a_0 b_1 - a_1 b_0$ $c_1 \leftarrow a_0 b_2 - a_2 b_0$ $c_2 \leftarrow a_1 b_2 - a_2 b_1$

The relative orientation of a point and a line and that of a line and a point are also given by the same formula, and the polar complements are the same:

plane $\leftarrow$ line $\vee$ point plane $\leftarrow$ point $\vee$ line vacuum $\leftarrow$ point $\wedge$ line vacuum $\leftarrow$ line $\wedge$ point sign $\leftarrow$ point $\diamond$ line sign $\leftarrow$ line $\diamond$ point
$c_0 \leftarrow a_0 b_2 - a_1 b_1 + a_2 b_0$

line $\leftarrow$ point$^{\vdash}$ line $\leftarrow$ point$^{\dashv}$ point $\leftarrow$ line$^{\vdash}$ point $\leftarrow$ line$^{\dashv}$
$c_0 \leftarrow a_3$ $c_1 \leftarrow -a_2$ $c_2 \leftarrow a_1$

5.3. Formulas for three-dimensional geometry

In three dimensions the intersting flats are points, lines, and planes. Unlike the previous cases, the join of two points and the meet of two planes are given by different formulas:

line ← point ∨ point
$c_0 \leftarrow a_0b_1 - a_1b_0$
$c_1 \leftarrow a_0b_2 - a_2b_0$
$c_2 \leftarrow a_1b_2 - a_2b_1$
$c_3 \leftarrow a_0b_3 - a_3b_0$
$c_4 \leftarrow a_1b_3 - a_3b_1$
$c_5 \leftarrow a_2b_3 - a_3b_2$

line ← plane ∧ plane
$c_0 \leftarrow a_0b_1 - a_1b_0$
$c_1 \leftarrow a_0b_2 - a_2b_0$
$c_2 \leftarrow a_0b_3 - a_3b_0$
$c_3 \leftarrow a_1b_2 - a_2b_1$
$c_4 \leftarrow a_1b_3 - a_3b_1$
$c_5 \leftarrow a_2b_3 - a_3b_2$

plane ← line ∨ point
$c_0 \leftarrow a_0b_2 - a_1b_1 + a_2b_0$
$c_1 \leftarrow a_0b_3 - a_3b_1 + a_4b_0$
$c_2 \leftarrow a_1b_3 - a_3b_2 + a_5b_0$
$c_3 \leftarrow a_2b_3 - a_4b_2 + a_5b_1$

point ← line ∧ plane
$c_0 \leftarrow a_0b_2 - a_1b_1 + a_3b_0$
$c_1 \leftarrow a_0b_3 - a_2b_1 + a_4b_0$
$c_2 \leftarrow a_1b_3 - a_2b_2 + a_5b_0$
$c_3 \leftarrow a_3b_3 - a_4b_2 + a_5b_1$

The relative orientation of a point and a plane and that of a plane and a point are also given by the same formulas, even though the operation is anticommutative. What happens is that the formula for $c \leftarrow a \vee b$ (a point, b plane) is antisymmetric in a and b. Therefore, interchanging a and b in the formula and negating everything gives back that same formula:

space ← plane ∨ point space ← point ∨ plane vacuum ← point ∧ plane vacuum ← plane ∧ point sign ← point ◇ plane sign ← plane ◇ point
$c_0 \leftarrow a_0b_3 - a_1b_2 + a_2b_1 - a_3b_0$

Here is how we compute the relative orientation of two lines:

space ← line ∨ line vacuum ← line ∧ line sign ← line ◇ line
$c_0 \leftarrow a_0b_5 - a_1b_4 + a_2b_3 + a_3b_2 - a_4b_1 + a_5b_0$

This being a space of odd dimension, the two polar complements are distinct, for points and planes, but the same for lines:

plane ← point$^\vdash$ point ← plane$^\vdash$
$c_0 \leftarrow -a_3$ $c_1 \leftarrow a_2$ $c_2 \leftarrow -a_1$ $c_3 \leftarrow a_0$

line ← line$^\vdash$ line ← line$^\dashv$
$c_0 \leftarrow a_5$ $c_1 \leftarrow -a_4$ $c_2 \leftarrow a_3$ $c_3 \leftarrow a_2$ $c_4 \leftarrow -a_1$ $c_5 \leftarrow a_0$

plane ← point$^\dashv$ point ← plane$^\dashv$
$c_0 \leftarrow a_3$ $c_1 \leftarrow -a_2$ $c_2 \leftarrow a_1$ $c_3 \leftarrow -a_0$

5.4. Four-dimensional geometry

In four-dimensional geometry the complexity of the formulas begins to get prohibitive, and there are few simplifying coincidences:

line ← point ∨ point
$c_0 \leftarrow a_0b_1 - a_1b_0$ $c_1 \leftarrow a_0b_2 - a_2b_0$ $c_2 \leftarrow a_1b_2 - a_2b_1$ $c_3 \leftarrow a_0b_3 - a_3b_0$ $c_4 \leftarrow a_1b_3 - a_3b_1$ $c_5 \leftarrow a_2b_3 - a_3b_2$ $c_6 \leftarrow a_0b_4 - a_4b_0$ $c_7 \leftarrow a_1b_4 - a_4b_1$ $c_8 \leftarrow a_2b_4 - a_4b_2$ $c_9 \leftarrow a_3b_4 - a_4b_3$

plane ← 3-space ∧ 3-space
$c_0 \leftarrow a_0b_1 - a_1b_0$ $c_1 \leftarrow a_0b_2 - a_2b_0$ $c_2 \leftarrow a_0b_3 - a_3b_0$ $c_3 \leftarrow a_0b_4 - a_4b_0$ $c_4 \leftarrow a_1b_2 - a_2b_1$ $c_5 \leftarrow a_1b_3 - a_3b_1$ $c_6 \leftarrow a_1b_4 - a_4b_1$ $c_7 \leftarrow a_2b_3 - a_3b_2$ $c_8 \leftarrow a_2b_4 - a_4b_2$ $c_9 \leftarrow a_3b_4 - a_4b_3$

plane ← point ∨ line
$c_0 \leftarrow a_0b_2 - a_1b_1 + a_2b_0$
$c_1 \leftarrow a_0b_4 - a_1b_3 + a_3b_0$
$c_2 \leftarrow a_0b_5 - a_2b_3 + a_3b_1$
$c_3 \leftarrow a_1b_5 - a_2b_4 + a_3b_2$
$c_4 \leftarrow a_0b_7 - a_1b_6 + a_4b_0$
$c_5 \leftarrow a_0b_8 - a_2b_6 + a_4b_1$
$c_6 \leftarrow a_1b_8 - a_2b_7 + a_4b_2$
$c_7 \leftarrow a_0b_9 - a_3b_6 + a_4b_3$
$c_8 \leftarrow a_1b_9 - a_3b_7 + a_4b_4$
$c_9 \leftarrow a_2b_9 - a_3b_8 + a_4b_5$

line ← 3-space ∧ plane
$c_0 \leftarrow a_0b_4 - a_1b_1 + a_2b_0$
$c_1 \leftarrow a_0b_5 - a_1b_2 + a_3b_0$
$c_2 \leftarrow a_0b_6 - a_1b_3 + a_4b_0$
$c_3 \leftarrow a_0b_7 - a_2b_2 + a_3b_1$
$c_4 \leftarrow a_0b_8 - a_2b_3 + a_4b_1$
$c_5 \leftarrow a_0b_9 - a_3b_3 + a_4b_2$
$c_6 \leftarrow a_1b_7 - a_2b_5 + a_3b_4$
$c_7 \leftarrow a_1b_8 - a_2b_6 + a_4b_4$
$c_8 \leftarrow a_1b_9 - a_3b_6 + a_4b_5$
$c_9 \leftarrow a_2b_9 - a_3b_8 + a_4b_7$

3-space ← point ∨ plane
$c_0 \leftarrow a_0b_3 - a_1b_2 + a_2b_1 - a_3b_0$
$c_1 \leftarrow a_0b_6 - a_1b_5 + a_2b_4 - a_4b_0$
$c_2 \leftarrow a_0b_8 - a_1b_7 + a_3b_4 - a_4b_1$
$c_3 \leftarrow a_0b_9 - a_2b_7 + a_3b_5 - a_4b_2$
$c_4 \leftarrow a_1b_9 - a_2b_8 + a_3b_6 - a_4b_3$

point ← 3-space ∧ line
$c_0 \leftarrow a_0b_6 - a_1b_3 + a_2b_1 - a_3b_0$
$c_1 \leftarrow a_0b_7 - a_1b_4 + a_2b_2 - a_4b_0$
$c_2 \leftarrow a_0b_8 - a_1b_5 + a_3b_2 - a_4b_1$
$c_3 \leftarrow a_0b_9 - a_2b_5 + a_3b_4 - a_4b_3$
$c_4 \leftarrow a_1b_9 - a_2b_8 + a_3b_7 - a_4b_6$

3-space ← plane ∨ point
$c_0 \leftarrow a_0b_3 - a_1b_2 + a_2b_1 - a_3b_0$
$c_1 \leftarrow a_0b_4 - a_4b_2 + a_5b_1 - a_6b_0$
$c_2 \leftarrow a_1b_4 - a_4b_3 + a_7b_1 - a_8b_0$
$c_3 \leftarrow a_2b_4 - a_5b_3 + a_7b_2 - a_9b_0$
$c_4 \leftarrow a_3b_4 - a_6b_3 + a_8b_2 - a_9b_1$

point ← line ∧ 3-space
$c_0 \leftarrow a_0b_3 - a_1b_2 + a_3b_1 - a_6b_0$
$c_1 \leftarrow a_0b_4 - a_2b_2 + a_4b_1 - a_7b_0$
$c_2 \leftarrow a_1b_4 - a_2b_3 + a_5b_1 - a_8b_0$
$c_3 \leftarrow a_3b_4 - a_4b_3 + a_5b_2 - a_9b_0$
$c_4 \leftarrow a_6b_4 - a_7b_3 + a_8b_2 - a_9b_1$

plane ← line ∨ point
$c_0 \leftarrow a_0b_2 - a_1b_1 + a_2b_0$
$c_1 \leftarrow a_0b_3 - a_3b_1 + a_4b_0$
$c_2 \leftarrow a_1b_3 - a_3b_2 + a_5b_0$
$c_3 \leftarrow a_2b_3 - a_4b_2 + a_5b_1$
$c_4 \leftarrow a_0b_4 - a_6b_1 + a_7b_0$
$c_5 \leftarrow a_1b_4 - a_6b_2 + a_8b_0$
$c_6 \leftarrow a_2b_4 - a_7b_2 + a_8b_1$
$c_7 \leftarrow a_3b_4 - a_6b_3 + a_9b_0$
$c_8 \leftarrow a_4b_4 - a_7b_3 + a_9b_1$
$c_9 \leftarrow a_5b_4 - a_8b_3 + a_9b_2$

line ← plane ∧ 3-space
$c_0 \leftarrow a_0b_2 - a_1b_1 + a_4b_0$
$c_1 \leftarrow a_0b_3 - a_2b_1 + a_5b_0$
$c_2 \leftarrow a_0b_4 - a_3b_1 + a_6b_0$
$c_3 \leftarrow a_1b_3 - a_2b_2 + a_7b_0$
$c_4 \leftarrow a_1b_4 - a_3b_2 + a_8b_0$
$c_5 \leftarrow a_2b_4 - a_3b_3 + a_9b_0$
$c_6 \leftarrow a_4b_3 - a_5b_2 + a_7b_1$
$c_7 \leftarrow a_4b_4 - a_6b_2 + a_8b_1$
$c_8 \leftarrow a_5b_4 - a_6b_3 + a_9b_1$
$c_9 \leftarrow a_7b_4 - a_8b_3 + a_9b_2$

3-space ← line ∨ line
$c_0 \leftarrow a_0b_5 - a_1b_4 + a_2b_3 + a_3b_2 - a_4b_1 + a_5b_0$
$c_1 \leftarrow a_0b_8 - a_1b_7 + a_2b_6 + a_6b_2 - a_7b_1 + a_8b_0$
$c_2 \leftarrow a_0b_9 - a_3b_7 + a_4b_6 + a_6b_4 - a_7b_3 + a_9b_0$
$c_3 \leftarrow a_1b_9 - a_3b_8 + a_5b_6 + a_6b_5 - a_8b_3 + a_9b_1$
$c_4 \leftarrow a_2b_9 - a_4b_8 + a_5b_7 + a_7b_5 - a_8b_4 + a_9b_2$

point ← plane ∧ plane
$c_0 \leftarrow a_0b_7 - a_1b_5 + a_2b_4 + a_4b_2 - a_5b_1 + a_7b_0$
$c_1 \leftarrow a_0b_8 - a_1b_6 + a_3b_4 + a_4b_3 - a_6b_1 + a_8b_0$
$c_2 \leftarrow a_0b_9 - a_2b_6 + a_3b_5 + a_5b_3 - a_6b_2 + a_9b_0$
$c_3 \leftarrow a_1b_9 - a_2b_8 + a_3b_7 + a_7b_3 - a_8b_2 + a_9b_1$
$c_4 \leftarrow a_4b_9 - a_5b_8 + a_6b_7 + a_7b_6 - a_8b_5 + a_9b_4$

The relative orientations of points vs. 3-space and 3-space vs. point are given by the same formula. Again, it is a case of both the formula and the operation being antisymmetric in a and b:

4-space ← 3-space ∨ point 4-space ← point ∨ 3-space vacuum ← point ∧ 3-space vacuum ← 3-space ∧ point sign ← point ◇ 3-space sign ← 3-space ◇ point
$c_0 \leftarrow a_0b_4 - a_1b_3 + a_2b_2 - a_3b_1 + a_4b_0$

On the other hand, the formulas for line vs. plane and plane vs. line are different, even though the operation itself is commutative:

4-space ← line ∨ plane vacuum ← line ∧ plane sign ← line ◇ plane
$c_0 \leftarrow a_0b_9 - a_1b_8 + a_2b_7 + a_3b_6 - a_4b_5 + a_5b_4 - a_6b_3 + a_7b_2 - a_8b_1 + a_9b_0$

4-space ← plane ∨ line vacuum ← plane ∧ line sign ← plane ◇ line
$c_0 \leftarrow a_0b_9 - a_1b_8 + a_2b_7 - a_3b_6 + a_4b_5 - a_5b_4 + a_6b_3 + a_7b_2 - a_8b_1 + a_9b_0$

The two polar complements coincide:

plane ← line$^\vdash$ plane ← line$^\dashv$
$c_0 \leftarrow a_9$
$c_1 \leftarrow -a_8$
$c_2 \leftarrow a_7$
$c_3 \leftarrow -a_6$
$c_4 \leftarrow a_5$
$c_5 \leftarrow -a_4$
$c_6 \leftarrow a_3$
$c_7 \leftarrow a_2$
$c_8 \leftarrow -a_1$
$c_9 \leftarrow a_0$

3-space ← point$^\vdash$ 3-space ← point$^\dashv$ point ← 3-space$^\vdash$ point ← 3-space$^\dashv$
$c_0 \leftarrow a_4$
$c_1 \leftarrow -a_3$
$c_2 \leftarrow a_2$
$c_3 \leftarrow -a_1$
$c_4 \leftarrow a_0$

line ← plane$^\vdash$ line ← plane$^\dashv$
$c_0 \leftarrow a_9$
$c_1 \leftarrow -a_8$
$c_2 \leftarrow a_7$
$c_3 \leftarrow a_6$
$c_4 \leftarrow -a_5$
$c_5 \leftarrow a_4$
$c_6 \leftarrow -a_3$
$c_7 \leftarrow a_2$
$c_8 \leftarrow -a_1$
$c_9 \leftarrow a_0$

Note that the formulas for join in $(\nu-1)$-dimensional geometry can be obtained from the ν-dimensional ones by dropping all terms that involve non-existing coordinates. It follows that procedures for two- and three-dimensional join can be coded so as to share substantial portions of their code.

If we want to do geometry in spaces of dimension higher than four, or of variable dimension, then this case-by-case approach is not feasible, and we have to write a single routine that computes formula (2) for all combinations of r, s, and n. However, in higher dimensions the Plücker coordinates themselves are impractical, and the simplex representation is more appropriate.

6. Projective maps in Plücker coordinates

We have seen that computing the image of a point by a projective map means post-multiplying its homogeneous coordinates by the associate matrix. What about flats of higher dimensions? The answer is rather straightforward:

Theorem 1. *The image of a flat* $[a_0, a_1 \ldots]^k$ *by a projective map* $M = [\![M]\!]$ *is the*

flat $[b_0, b_1, \ldots]^k$ *where*

$$b_j = \sum_i a_i (M^{(k)})^i_j \tag{10}$$

and $M^{(k)}$ *is the* $\binom{n}{k} \times \binom{n}{k}$ *matrix whose elements are all* $k \times k$ *minor determinants of* M*, in natural order of row and column labels.*

PROOF: Consider a κ-dimensional flat a with coordinates $[a_0, a_1, \ldots]^k$. Let $u = (u^0; .. u^\kappa)$ be a representative basis of a in the homogeneous model (so that $a = <u^0, .. u^\kappa>$), viewed as a $k \times n$ matrix. Recall that a_i is the coordinate whose label is the integer set $I = k{:}i$, that is, the determinant of the $k \times k$ minor formed by taking the columns of u whose indices are in the set I.

Let $aM = b = [b_0, .. b_{m-1}]^k$. A representative simplex of b is $v = (v^0; .. v^\kappa)$ where $v^i_j = \sum_t u^i_t M^t_j$. Therefore, the coordinate with label set J is given by the determinant

$$b_J = \begin{vmatrix} v^0_{j_0} & v^0_{j_1} & \cdots & v^0_{j_\kappa} \\ v^1_{j_0} & v^1_{j_1} & \cdots & v^1_{j_\kappa} \\ \vdots & & & \vdots \\ v^\kappa_{j_0} & v^\kappa_{j_1} & \cdots & v^\kappa_{j_\kappa} \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{i_0} u^0_{i_0} M^{i_0}_{j_0} & \sum_{i_1} u^0_{i_1} M^{i_1}_{j_1} & \cdots & \sum_{i_\kappa} u^0_{i_\kappa} M^{i_\kappa}_{j_\kappa} \\ \sum_{i_0} u^1_{i_0} M^{i_0}_{j_0} & \sum_{i_1} u^1_{i_1} M^{i_1}_{j_1} & \cdots & \sum_{i_\kappa} u^1_{i_\kappa} M^{i_\kappa}_{j_\kappa} \\ \vdots & & & \vdots \\ \sum_{i_0} u^\kappa_{i_0} M^{i_0}_{j_0} & \sum_{i_1} u^\kappa_{i_1} M^{i_1}_{j_1} & \cdots & \sum_{i_\kappa} u^\kappa_{i_\kappa} M^{i_\kappa}_{j_\kappa} \end{vmatrix} \tag{11}$$

where $j_0, j_1, .. j_\kappa$ are the elements of J in increasing order, and each i_k ranges from 0 to ν. Note that each column of (11) is a linear combination of columns of the u matrix. Since the determinant of a matrix is a multilinear function of its columns, we can expand equation (11) into

$$b_J = \sum_{i_0} \sum_{i_1} \cdots \sum_{i_\kappa} \begin{vmatrix} u^0_{i_0} & u^0_{i_1} & \cdots & u^0_{i_\kappa} \\ u^1_{i_0} & u^1_{i_1} & \cdots & u^1_{i_\kappa} \\ \vdots & & & \vdots \\ u^\kappa_{i_0} & u^\kappa_{i_1} & \cdots & u^\kappa_{i_\kappa} \end{vmatrix} \cdot M^{i_0}_{j_0} M^{i_1}_{j_1} \cdots M^{i_\kappa}_{j_\kappa} \tag{12}$$

where the i_k still range independently over $0 \,..\, \nu$.

In summation (12), any term with two or more equal i's is zero. Moreover, two sequences $i_0 \ldots i_\kappa$ which differ only on the order of the elements will give rise to two terms that differ at most in their signs. More precisely, formula (12) is equivalent to

$$b_J = \sum_{0 \le i_0 < i_1 < \cdots < i_\kappa \le \nu} \begin{vmatrix} u^0_{i_0} & u^0_{i_1} & \cdots & u^0_{i_\kappa} \\ u^1_{i_0} & u^1_{i_1} & \cdots & u^1_{i_\kappa} \\ \vdots & & & \vdots \\ u^\kappa_{i_0} & u^\kappa_{i_1} & \cdots & u^\kappa_{i_\kappa} \end{vmatrix} \cdot \sum_\pi (-1)^{\|\pi\|} \cdot M^{i_{\pi(0)}}_{j_0} M^{i_{\pi(1)}}_{j_1} \cdots M^{i_{\pi(\kappa)}}_{j_\kappa}$$

where π ranges over all permutations of $0..\kappa$, and $\|\pi\|$ is the number of inversions in π. But the second summation is simply the minor determinant formed by lines $j_0 \ldots j_\kappa$ and columns $i_0 \ldots i_\kappa$ of matrix M:

$$b_J = \sum_{0 \le i_0 < i_1 < \cdots < i_\kappa \le \nu} \begin{vmatrix} u^0_{i_0} & u^0_{i_1} & \cdots & u^0_{i_\kappa} \\ u^1_{i_0} & u^1_{i_1} & \cdots & u^1_{i_\kappa} \\ \vdots & & & \vdots \\ u^\kappa_{i_0} & u^\kappa_{i_1} & \cdots & u^\kappa_{i_\kappa} \end{vmatrix} \cdot \begin{vmatrix} M^{i_0}_{j_0} & M^{i_1}_{j_0} & \cdots & M^{i_\kappa}_{j_0} \\ M^{i_0}_{j_1} & M^{i_1}_{j_1} & \cdots & M^{i_\kappa}_{j_1} \\ \vdots & & & \vdots \\ M^{i_0}_{j_\kappa} & M^{i_1}_{j_\kappa} & \cdots & M^{i_\kappa}_{j_\kappa} \end{vmatrix}.$$

Therefore, we conclude

$$b_J = \sum_I a_I (M^{(k)})^I_J$$

where I ranges over all k-element subsets of $0 \,..\, \nu$.

QED.

For example, if M is the matrix

$$\begin{pmatrix} 1 & 3 & 5 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 2 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

then $M^{(2)}$ is the matrix

$$M^{(2)} = \begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 0 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 4 \\ 0 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -10 & -5 & 0 & 5 & 10 \\ -1 & -5 & -10 & 3 & 10 & 20 \\ 3 & 0 & 0 & 6 & 0 & 0 \\ -3 & -5 & 0 & 0 & 3 & 5 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & -3 & 2 & 0 \end{pmatrix}.$$

This result is of little practical value, since computing the matrix $M^{(k)}$ and applying it is much too expensive. In general it is far more efficient to use the simplex representation (reduced or not), and map each vertex of the simplex through the original matrix M. This takes $O(kn^2)$ time, including the $O(k^2n)$ cost of putting the mapped simplex in reduced form. Even if the given flat and the answer are represented by Plücker coordinates, it is more efficient in general to find a representative simplex, map its vertices, and compute the Plücker coordinates of the result by the join formula.

This is already true even in the simplest non-trivial case, namely lines in three-space ($n = 3, k = 2$). Computing the matrix $M^{(2)}$ costs 72 multiplications (μ) and 36 additions/subtractions (α). Once we have this matrix, the cost of mapping a line is $36\mu + 30\alpha$. If instead we compute two independent points from the Plücker coordinates, map those points through M, and compute the join of the images, the

total cost will be about $8(3\mu+2\alpha)+6(\mu+2\alpha) = 30\mu+28\alpha$, plus some logic overhead. (Note that the representative points computed as described in secion 4.3 have only three non-zero coordinates). By comparison, mapping an arbitrary pair of points (an unreduced simplex) through the map M costs $32\mu + 24\alpha$. Mapping a reduced simplex and reducing the result costs in general $4(2\mu+2\alpha)+10\mu+5\alpha = 18\mu+13\alpha$. In spaces of higher dimensions Plücker coordinates are even more expensive, because the size of the matrix $M^{(k)}$, $\binom{n}{k} \times \binom{n}{k}$, grows exponentially with n and k.

7. Directions and parallelism

The direction of a flat is easily computed from its Plücker coordinates. Recall that the coordinates of Ω are $[0,..0,1]^{n-1}$. Therefore, the direction of a flat a of rank r is the flat of rank $r-1$ whose coordinates are

$$\begin{aligned}
(\operatorname{dir} a)_K &= (a \wedge \Omega)_K \\
&= \sum_{\substack{I \cap J = K \\ I \cup J = \{0..\nu\} \\ |I|=r \\ |J|=n-1}} (-1)^{|\bar{J}>\bar{I}|}\, a_I\, \Omega_J \\
&= \sum_{\substack{I \cap \{1..\nu\} = K \\ I \cup \{1..\nu\} = \{0..\nu\} \\ |I|=r}} (-1)^{|\{0\}>\bar{I}|}\, a_I \\
&= \begin{cases} 0 & \text{if } 0 \in K, \\ a_{0 \cup K} & \text{if } 0 \notin K. \end{cases}
\end{aligned} \tag{13}$$

where $0 \cup K$ is a shorthand for $\{0\} \cup K$. For example, the direction of a line l in three-space is the point x with coordinates

$$x_{\{0\}} = 0, \quad x_{\{1\}} = l_{\{01\}}, \quad x_{\{2\}} = l_{\{02\}}, \quad x_{\{3\}} = l_{\{03\}}$$

or, in positional notation, $\operatorname{dir}[l_0,..l_5]^2 = [0,\ l_0,\ l_1,\ l_3]$. In the same way we get formulas for the direction of a line of $\mathbf{T}_2$,

$$\operatorname{dir}[l_0,\ l_1,\ l_2]^2 = [0,\ l_0,\ l_1]^1$$

and a plane of $\mathbf{T}_3$:

$$\operatorname{dir}[h_0,\ h_1,\ h_2,\ h_3]^3 = [0,\ 0,\ h_0,\ 0,\ h_1,\ h_2]^2.$$

7.1. Parallelism

Recall that $a \Uparrow b$ was defined as a shorthand for $\operatorname{dir} a = \operatorname{dir} b$. From formula (13), we see that $a \Uparrow b$ if and only if the Plücker coordinates of a whose label set includes 0 are a positive multiple of the corresponding coordinates of b. For example, if a and b are two lines in three-space, we have $[a_0, .. a_5]^2 \Uparrow [b_0, .. b_5]^2$ if and only if

$$a_0 = \lambda b_0 \quad \text{and} \quad a_1 = \lambda b_1 \quad \text{and} \quad a_3 = \lambda b_3$$

for some $\lambda > 0$. If f is a proper flat of $\mathbf{T}_\nu$, and p is a point on the front range, then the flat passing through p and with same rank and direction as f has coordinates given by the formula

$$\begin{aligned}
(p \vee \operatorname{dir} f)_K &= \sum_{\substack{I \cup J = K \\ I \cap J = \emptyset \\ |I|=1,\, |J|=r-1}} (-1)^{|I>J|}\, p_I\, (\operatorname{dir} f)_J \\
&= \sum_{i \in K} (-1)^{|i>K\setminus i|}\, p_i\, (\operatorname{dir} f)_{K\setminus i} \\
&= \sum_{i \in K} (-1)^{|i>K\setminus i|}\, p_i \cdot \begin{cases} 0 & \text{if } 0 \in K \setminus i \\ f_{0 \cup K\setminus i} & \text{if } 0 \notin K \setminus i \end{cases} \\
&= \begin{cases} p_0 f_K & \text{if } 0 \in K, \\ \sum_{i \in K} (-1)^{|i>K\setminus i|}\, p_i\, f_{0 \cup K\setminus i} & \text{if } 0 \notin K. \end{cases}
\end{aligned}$$

For example, in three-space the line through $p = [p_0, .. p_3]$ and parallel to the line $l = [l_0, .. l_5]^2$ is

$$[p_0 l_0,\ p_0 l_1,\ p_1 l_1 - p_2 l_0,\ p_0 l_3,\ p_1 l_3 - p_3 l_0,\ p_2 l_3 - p_3 l_1]^2. \tag{14}$$

The analogous formula for two-dimensional geometry is obtained by dropping the last three coordinates, that is,

$$[p_0 l_0,\ p_0 l_1,\ p_1 l_1 - p_2 l_0]^2$$

The plane through p parallel to $h = [h_0, .. h_3]^3$ is

$$[p_0 h_0,\ p_1 h_1,\ p_2 h_2,\ p_1 h_2 - p_2 h_1 + p_3 h_0]^3.$$

Bibliography

Bracketed superscripts in the text (such as [3]) refer to the partial bibliographies at the end of each chapter. The listing below combines those partial bibliographies in a single alphabetical order.

- C. F. Adler:
  Modern geometry (2nd. ed.).
  McGraw-Hill (1970).

- M. Barnabei, A. Brini, and G.-C. Rota:
  On the exterior calculus of invariant theory.
  Journal of Algebra vol. 96 no. 1 (Sept. 1985), pp. 120–160.

- G. Berman:
  The wedge product.
  American Mathematical Monthly vol. 68 no. 2 (1961) 112–119.

- L. M. Blumenthal and K. Menger:
  Studies in geometry
  W. H. Freeman (1970).

- H. S. M. Coxeter:
  The real projective Plane.
  McGraw-Hill (1949).

- H. Crapo and J. Ryan:
  Scene analysis and geometric homology.
  Proceedings of the 2nd ACM Symposium on
  Computational Geometry (June 1986), 125–132.

- G. H. Golub and C. F. Van Loan:
  Matrix computations.
  Johns Hopkins University Press, Baltimore (1983).

- P. Hanrahan:
  A homogeneous geometry calculator.
  Technical memo 3-D no. 7, Computer Graphics Laboratory,
  New York Inst. of Technology (September 1984).

- D. Hestenes and G. Sobczyk:
  Clifford algebra to geometric calculus:
  A unified language for mathematics and physics.
  D. Reidel, Dordrecht (1984).

- W. V. D. Hodge, D. Pedoe:
  Methods of algebraic geometry.
  Cambridge University Press (1952).

- A. Inselberg and M. Reif:
  Convexity in parallel coordinates.
  [Manuscript] CS Department, Univ. of California at Los Angeles (June 1984).

- P. S. Modenov and A. S. Parkhomenko:
  Geometric transformations. vol. II: Projective transformations.
  Academic Press (1965).

- A. Penna and R. Patterson:
  Projective geometry and its
  applications to computer graphics.
  Prentice-Hall (1986).

- F. P. Preparata and D. E. Muller:
  Finding the intersection of n half-spaces in time $O(n \log n)$.
  Theoretical Computer Science vol. 8 (1979), pp. 44–55

- W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling:
  Numerical recipes: The art of scientific computing.
  Cambridge University Press, Cambridge (1986).

- R. F. Riesenfeld:
  Homogeneous coordinates and
  projective planes in computer graphics.
  IEEE Computer Graphics and Automation (Jan. 1981), pp. 50–55.

List of symbols

Symbol	Meaning	Page
$\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$	naturals, integers, reals	
$\mathbf{R}^\nu$	ν-dimensional real vector space	
$\mathbf{E}_\nu$	ν-dimensional Euclidean space	
$\mathbf{P}_\nu$	ν-dimensional projective space	
$\mathbf{S}_\nu$	ν-dimensional sphere of $\mathbf{R}^{\nu+1}$	13
Ω	line at infinity	2, 14
$\mathbf{T}_\nu$	canonical ν-dimensional two-sided space	13
Ω_ν	hyperplane at infinity of $\mathbf{P}_\nu$ or $\mathbf{T}_\nu$	2
Υ_ν	positive universe of $\mathbf{T}_\nu$	13, 19
$\mathcal{F}_\nu$, $\mathcal{F}_\nu^k$	all flats of $\mathbf{T}_\nu$, and all those with rank k	13, 19, 20
$d\infty$	point at infinity in direction d	2, 14
$\neg a$	opposite of flat a	19
$\sigma \circ a$	a if $\sigma = +1$, $\neg a$ if $\sigma = -1$	20
Λ	vacuum	20
rank(a)	rank of flat a	20
corank(a)	complementary rank of flat a	20
sign(x)	sign of the real number x	66
$(x^0; .. x^\mu)$	simplex with vertices $x^0, .. x^\mu$, or matrix with rows $x^0, .. x^\mu$	29, 31
$(\mathbf{e}^0; .. \mathbf{e}^\nu)$	canonical basis of $\mathbf{R}^n$, or standard simplex of $\mathbf{T}_\nu$	34
O	point $[1, 0, \ldots, 0]$, the front origin of $\mathbf{T}_\nu$	34
$a \vee b$	join of flats a and b	39
$a \wedge b$	meet (oriented intersection) of flats a and b	47
$a \diamond b$	relative orientation of a and b	59
$[x_0, .. x_\nu]$	the point of $\mathbf{T}_\nu$ with homogeneous coordinates $x_0, .. x_\nu$	16
$\langle h^0, .. h^\nu \rangle$	the hyperplane of $\mathbf{T}_\nu$ with coefficients $h^0, .. h^\nu$	66
$\mathbf{0}^k$	null object with rank k	44
$[\![M]\!]$	the projective map induced by the linear map M	68
$\overleftarrow{F}$	functional inverse of F	69
xF	image of x by function F	69
I_A	identity function on set A	69
FG	composition of functions F and G, in that order	69
$\mathcal{M}_\nu$	projective maps of $\mathbf{T}_\nu$ to itself	70

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SRC Reports

"A Kernel Language for Modules and Abstract Data Types."
R. Burstall and B. Lampson.
Research Report 1, September 1, 1984.

"Optimal Point Location in a Monotone Subdivision."
Herbert Edelsbrunner, Leo J. Guibas, and Jorge Stolfi.
Research Report 2, October 25, 1984.

"On Extending Modula-2 for Building Large, Integrated Systems."
Paul Rovner, Roy Levin, John Wick.
Research Report 3, January 11, 1985.

"Eliminating **go to**'s while Preserving Program Structure."
Lyle Ramshaw.
Research Report 4, July 15, 1985.

"Larch in Five Easy Pieces."
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Research Report 5, July 24, 1985.

"A Caching File System for a Programmer's Workstation."
Michael D. Schroeder, David K. Gifford, and Roger M. Needham.
Research Report 6, October 19, 1985.

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Research Report 7, November 14, 1985.

"On Interprocess Communication."
Leslie Lamport.
Research Report 8, December 25, 1985.

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Research Report 9, April 1, 1986.

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Research Report 10, May 1, 1986.

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Research Report 12, June 23, 1986.

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Research Report 13, August 20, 1986.

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Research Report 14, November 27, 1986.

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Research Report 15, December 25, 1986. Revised January 26, 1988

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Research Report 16, April 2, 1987.

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Research Report 23, December 30, 1987.

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Andrew D. Birrell, Michael B. Jones, and Edward P. Wobber.
Research Report 24, January 30, 1988.

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Research Report 25, February 14, 1988.

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Research Report 28, May 1, 1988.

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Research Report 30, August 15, 1988.

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Research Report 31, August 25, 1988.

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Research Report 32, October 15, 1988.

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Research Report 33, November 1, 1988.

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Research Report 34, January 2, 1989.

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Research Report 35, January 6, 1989.

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